Affine Term Structure Models with Stochastic Lower Bound: An application to Euro-Area OIS Rates

12th FINANCIAL RISKS INTERNATIONAL FORUM: Low Interest rate environement

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Motivation: from Zero Lower Bound (ZLB) to time-varying Lower Bound (LB)

- Since July 2012, the EONIA reached zero and stayed there until June 2014. From June 2014, the EONIA followed the Deposit Facility Rate (DFR) of the European Central Bank into negative territory $\Rightarrow$ ZLB became a **time-varying** (possibly negative) Lower Bound.

In this context,

ZLB term structure models are not appropriate anymore.
We need a model where we have:

(i) Persistence of the short rate at the LB for an extended period of time, while longer maturities rates continue varying,

(ii) Short rate that can take negative value,

(iii) Time-varying (possibly negative) LB

Literature review

Existing models already account for (i) and (ii):

- Lemke, Vladu (2016):
  - ✔ (i) Persistence of the short rate and (ii) negative LB;
  - ✘ shadow rate less flexible than affine model (no closed form formulas for rates and constant conditional volatilities),
    piecewise constant non-stochastic LB
- Geiger, Schupp:
  - ✔ (i) Persistence of the short rate and (ii) negative LB;
  - ✘ shadow rate less flexible than affine model (no closed form formulas for rates and constant conditional volatilities),
    non-stochastic LB
- Wu, Xia (2018):
  - ✔ (i) Persistence of the short rate and (ii) stochastic negative LB;
  - ✘ shadow rate less flexible than affine model (no closed form formulas for rates and constant conditional volatilities),
    Bonds are priced as if the current LB would prevail in the future.

This paper

New term structure model with a **Stochastic Lower Bound** (SLB) which can take negative values:

- Model features:
  - short rate can stay at the SLB while the other rates are varying with specific volatilities
  - affine closed-form formulas
  - Q.M.L. Estimation via Kalman Filter
- Outputs
  - **nominal rate decomposition** $\rightarrow$ expectation $+$ term premium,
  - **time-varying probabilities** of additional moves of the LB,
  - **distribution of lift-off dates**,

    On going work:
  - **Impulse Response Function** (IRF) of shocks on the LB,
  - **Scenario Response Distribution** (SRD) of a specific future path of the LB.
- Results overview
  - very good fit, even when negative, of observed nominal rates for any maturities,
  - evidence of decreasing Term Premia since 2014,
  - distribution of possible lift-off dates.

The Model: SLB

We define the **Stochastic Lower Bound (SLB)** as:

$$SLB_t \propto -Y_t, \text{ multiple of 10 b.p.s.}$$

such that $\Delta Y_t = Y_t - Y_{t-1}$ is the sum of two opposite forces:

$$\Delta Y_t = \Delta N_t^+ - \Delta N_t^- \in \mathbb{N}, \text{ with:}$$

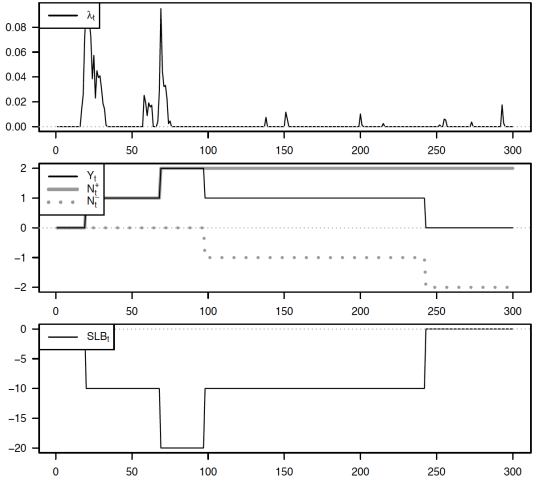
$\Delta N_t^+ \mid \underline{\lambda_t}, \underline{\Delta N_{t-1}^+}, \underline{\Delta N_{t-1}^-} \sim \mathcal{P}(\lambda_t)$

- Positive force
- Responsible for the **decreases** of SLB_t
- Driven by an an intensity process $\lambda_t \sim ARG_0\left(\tilde{\alpha}, \tilde{\beta}, \tilde{\mu}\right) \geq 0$ ▸ ARG0

$\Delta N_t^- \mid \underline{\lambda_t}, \underline{\Delta N_t^+}, \underline{\Delta N_{t-1}^-} \sim \mathcal{P}(\eta Y_{t-1})$

- Return force
- Responsible for the **increases** of SLB_t

Simulation of the Stochastic Lower Bound (SLB)



The Model: the global process

We introduce a 4-dimensional VARG process: $X_t = (X_{1,t}, X_{2,t}, X_{3,t}, X_{4,t})'$

- the $X_{j,t}$'s are positive
- $X_{1,t}, X_{2,t}$ can stay at zero (ARG_0 processes)
- $X_{3,t}, X_{4,t}$ (ARG_ν processes) cause $X_{1,t}, X_{2,t}$

X_t independent of $\tilde{X}_t = (\lambda_t, Y_t)'$, dedicated to the SLB modelling.

global process $Z_t = \left(X_t', \tilde{X}_t'\right)'$ is affine

Short rate can stay at SLB_t:

$$
\begin{aligned}
r_t &= SLB_t + X_{1,t} + X_{2,t} && (1)\\
&= \delta' Z_t && (2)
\end{aligned}
$$

Stochastic Discount Factor (SDF)

The SDF $\Rightarrow$ defines the risk adjustment between the historical and the risk-neutral world.
$\Rightarrow$ is exponential affine.

$$M_{t,t+1} = \exp\left(\theta' Z_{t+1} - r_t - \psi_0^Z(\theta) - \psi_1^Z(\theta)' Z_t\right). \tag{3}$$

- the risk-neutral dynamics is also affine
- the risk-neutral dynamics of X_t is still a VARG, with the same causality structure

The Pricing

Let us denote by $R_{t,h}$ the interest rate at time t for maturity h:

$$R_{t,h} = -\frac{1}{h} \ln \mathbb{E}_t^Q \left[\exp(-\delta' Z_t - \ldots - \delta' Z_{t+h-1} \right] .$$

Due to multi-horizon conditional Laplace transform of affine models we get an **affine represention of the yield**:

$$R_{t,h} = \bar{B}_h + \bar{A}_h' Z_t$$

- $X_{3,t}$, $X_{4,t}$ cause $X_{1,t}$, $X_{2,t}$,
- $R_{t,h}$ is a function of **all** the $X_{i,t}$'s,

$\Rightarrow$ $R_{t,h}$ can move even if $X_{1,t} = X_{2,t} = 0$ (i.e. even if the short rate stays at the lower bound)

State-Space representation

Since Z_t is affine under $\mathbb{P}$, we have:

$$Z_{t+1} = m_Z + M_Z Z_t + \Sigma_Z^{1/2}(Z_t)\epsilon_{t+1} ,$$

with:

- ϵ_t, a martingale difference with zero mean and identity var-covar matrix,
- $\Sigma_Z(Z_t)$ is affine in Z_t,
- M_Z and $\Sigma_Z^{1/2}(Z_t)$ are block diagonal.

The affine representation can hold for all observable variables $Obs_t = [R_t', S_t', V_t', SLB_t]'$ we consider:

- R_t: nominal rates (7 obs.)
- S_t: Forecasts (6 obs.)
- V_t: Volatilities (2 obs.)
- SLB_t: observed Lower bound (1 obs.)

$\Rightarrow$**State-space representation**:

$$\left\{ \begin{array}{ll} Z_{t+1} & = m_Z + M_Z Z_t + \Sigma_Z^{1/2}(Z_t)\,\epsilon_{t+1} \\ Obs_t & = \Gamma_0 + \Gamma_1 Z_t + \Omega\, U_t \end{array} \right. \tag{4}$$

The estimation of the model is done by (quasi) maximum likelihood via Kalman Filter

Predictions

From transition equation we have:

$$Z_{t+1} = m_Z + M_Z Z_t + u_{t+1} \tag{5}$$

Iterating pervious equation we get:

$$Z_{t+k} = \sum_{i=0}^{k-1} M_Z^i \, m_Z + M_Z^k \, Z_t + \sum_{i=0}^{k-1} M_Z^i \, u_{t+k-i}. \tag{6}$$

We can compute conditional mean and variance of the factors:

$$E_t\left[Z_{t+k}\right] = \sum_{i=0}^{k-1} M_Z^i \, m_Z + M_Z^k \, Z_t$$

$$V_t\left[Z_{t+k}\right] = \sum_{i=0}^{k-1} M_Z^i \Sigma_Z \left(E_t\left[Z_{t+k-i-1}\right]\right) M_Z^{i\,\prime}$$

Since $R_{t,h} = \bar{B}_h + \bar{A}_h' Z_t$:

- $E_t\left[R_{t+k,h}\right] = \bar{B}_h + \bar{A}_h' E_t\left[Z_{t+k}\right]$, affine in Z_t
- $V_t\left[R_{t+k,h}\right] = \bar{A}_h' V_t\left[Z_{t+k}\right] \bar{A}_h$, affine in Z_t

Impulse Response Functions

We can also compute Impulse Response Function (IRF) of the expectation of $R_{t+k,h}$ to a shock changing $Z_t = z_t$ into $Z_t = z_t + q$ by:

$$\begin{aligned} IRF^{E}_{h,q}(k) &= E\left[R_{t+k,h} \mid Z_t = z_t + q\right] - E\left[R_{t+k,h} \mid Z_t = z_t\right] \\ &= \bar{A}'_h M_Z^k q \end{aligned}$$

Similarly for the variance we have as IRF:

$$\begin{aligned} IRF^{V}_{h,q}(k) &= V\left[R_{t+k,h} \mid Z_t = z_t + q\right] - V\left[R_{t+k,h} \mid Z_t = z_t\right] \\ &= \bar{A}'_h \sum_{i=0}^{k-1} M_Z^i \, \Sigma\left(M_Z^{k-i-1} q\right) M_Z^{i\,\prime} \bar{A}_h \end{aligned}$$

We thus have closed-form formulas:

- to directly estimate the effect of a shock on the SLB at t for any horizon k
- to evaluate policy decisions from central banks

Scenarios Response Distribution

Evaluate the impact on any future value of $R_{t,h}$ of a forward guidance imposing a full path $Y_{t+1} = y_{t+1}, \ldots, Y_{t+k} = y_{t+k}$.

$\Rightarrow$ Solved if we are able to simulate in the conditional distribution of

$$\left(\lambda_{t+1,t+k} \mid \tilde{X}_t, Y_{t+1,t+k} = y_{t+1,t+k}\right)$$

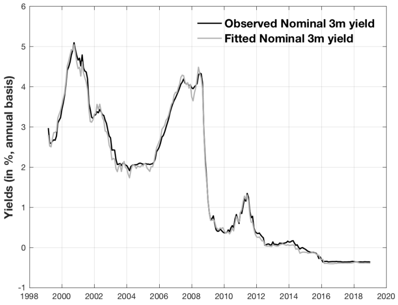
Metropolis-Hastings

Empirical analysis: Data

Obs_t:

- $\mathbf{R_t}$: EONIA Overnight Index Swaps (OIS) rates 3-, 6-month,1-, 2-, 5-, 7- and 10-year maturities, monthly frequency from January 1999 to January 2019.
- $\mathbf{S_t}$
  - forecast on LB build from 3-month Euribor yield forecast, 3 months ahead,
  - short-term forecasts on a long-term yield: consensus forecasts of the 10-year German Bond yield, 3 and 12 months ahead,
  - long-term forecasts of short-term yield:
    - consensus forecast of the 3-month Euribor yield, 5 years ahead,
    - mean of the consensus forecasts of the 3-month Euribor between 1 and 10 years ahead.
- $\mathbf{V_t}$: volatilities are computed on daily data using 3-months rolling windows.
- $\mathbf{SLB_t}$: the observed lower bound is 0 before June 2014, and then follows the DFR when negative.

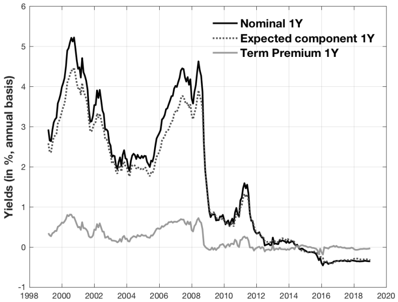
Empirical analysis: 3-month and 10-year observed vs fitted nominal yields



(a)

(b)

Empirical analysis: 1- and 10-year Nominal rates decomposition



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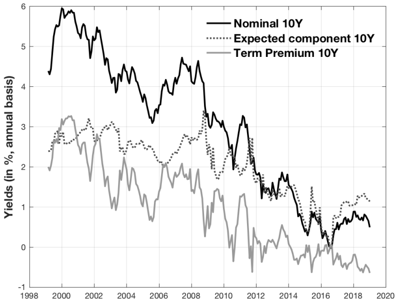
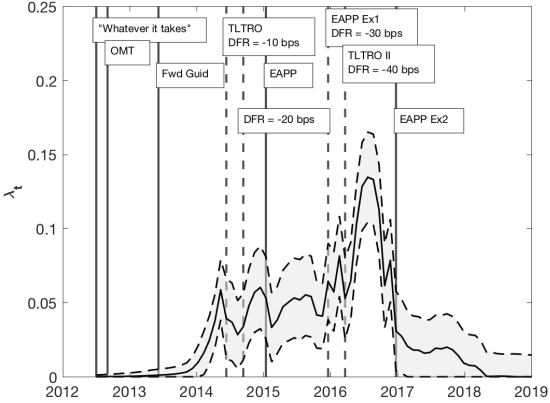


(d)

Empirical analysis: λ_t factor vs ECB communication



Empirical analysis: Lift-off quantiles vs ECB communication

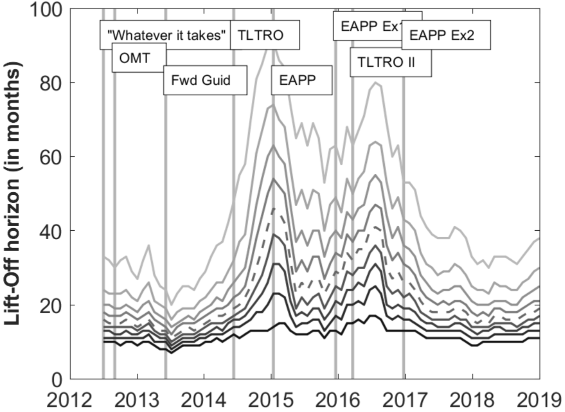


Figure: 10 to 90 percentile on lift-off dates for 3-month yield with a lift-off threshold of + 30 bps

APPENDIX

Definitions

Definition: Non central Gamma distribution

Let X be a non-negative random variable. $X \sim \gamma_\nu(\lambda, \mu)$, if $X \mid W \sim \gamma_{\nu+W}(\mu)$, with $W \sim \mathcal{P}(\lambda)$
The distribution becomes a non central gamma-zero one if $\nu = 0$

Definition: ARG_0

$(X_t \mid \underline{X_{t-1}})$ follows a Auto-Regressive Gamma Zero process $ARG_{\nu=0}(\alpha, \beta, \mu)$, if $(X_t \mid \underline{X_{t-1}}) \sim \gamma_0\left(\alpha + \beta X_t, \mu\right)$

Definition: $VARG$

$(X_t \mid \underline{X_{t-1}})$ follows a Vector Auto-Regressive Gamma process $VARG$, if its scalar components $(X_{j,t} \mid \underline{X_{t-1}})$ are independent conditional on $\underline{X_{t-1}}$ and their conditional distribution is the Gamma distribution

$$(X_{j,t} \mid \underline{X_{t-1}}) \sim \gamma_{\nu_j}\left(\alpha_j + \beta_j' X_{t-1}, \mu_j\right), \quad j \in \{1, \ldots, 4\}.$$

$\alpha_j \geq 0,\ \mu_j > 0,\ \beta_j > 0, \nu_j \geq 0$
NB: If $\nu_j = 0$, $X_{j,t}$ **can stay at 0**.

back

Metropolis-Hastings

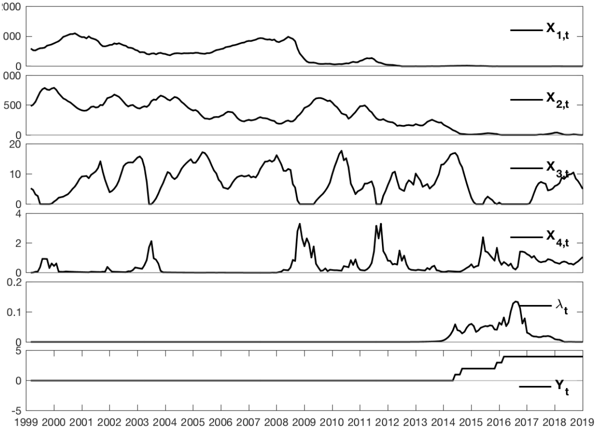
The p.d.f. of this conditional distribution is **proportional to** the joint p.d.f. of $(\lambda_{t+1,t+k}, Y_{t+1,t+k})$ given $\tilde{X}_t$ evaluated at $y_{t+1,t+k}$, that is to say:

$$\prod_{\tau=t+1}^{t+k} f\left(\lambda_\tau \mid \lambda_{\tau-1}\right) p\left(y_\tau \mid \lambda_\tau, y_{\tau-1}\right) \tag{7}$$

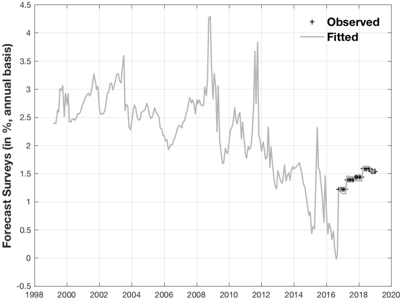
We can use a **Metropolis-Hastings** (MH) algorithm here to simulate and obtain drawings in the conditional distribution of $Z_{t+1,t+k} \mid Z_t, Y_{t+1,t+k} = y_{t+1,t+k}$.
From these drawings we obtain drawings in the distribution of $R_{t+1,h}, \ldots, R_{t+k,h}$ given Z_t and $Y_{t+1,t+k} = y_{t+1,t+k}$, which allow to estimate associated conditional distributions, moments, quantiles under the scenario of the studied path of the SLB.

back

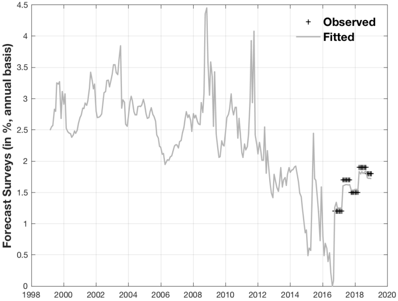
Model factors



Empirical analysis: 3-month Euribor in 5-year and 3-month Euribor in 1- to 10-year consensus forecasts



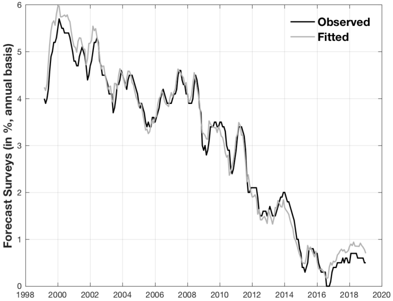
(a)



(b)

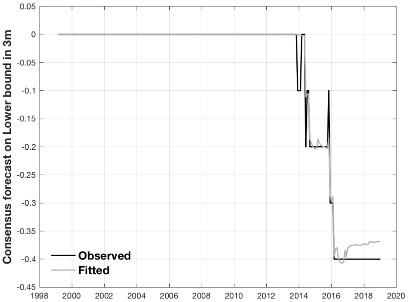
Empirical analysis: 10-year German Bund in 3-month and 10-year German Bund in 1-year consensus forecasts

(c)



(d)

Empirical analysis: Lower Bound forecast in 3-month and in 1-year



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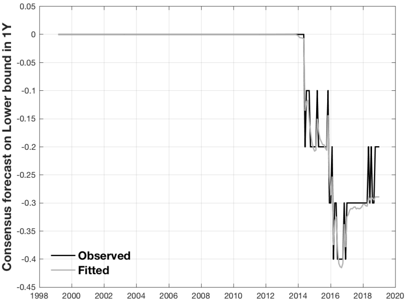
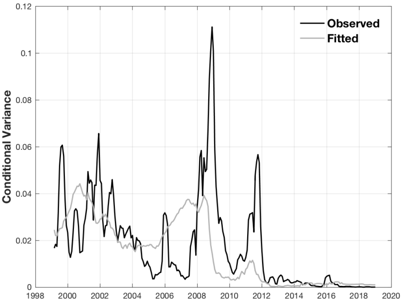


(f)

Empirical analysis: 1- and 10-year rates Volatilities



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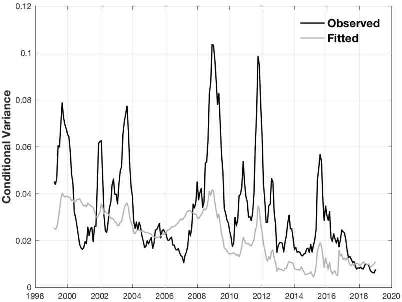

(h)