Multiple Curve Lévy Forward Price Model Allowing For Negative Interest Rates

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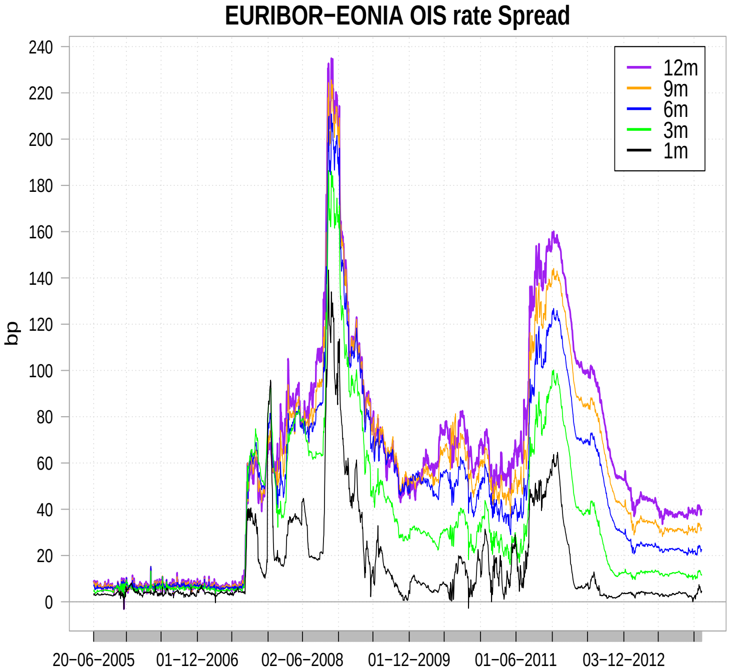

EURIBOR−EONIA OIS rate Spread
12m
9m
6m
3m
1m
bp
240
220
200
180
160
140
120
100
80
60
40
20
0
20−06−2005
01−12−2006
02−06−2008
01−12−2009
01−06−2011
03−12−2012

Basic interest rates

$B(t,T)$: price at time $t \in [0, T]$ of a default-free zero coupon bond

$f(t,T)$: instantaneous forward rate (short rate $r(t) = f(t,t)$)

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\,\mathrm{d}u\right)$$

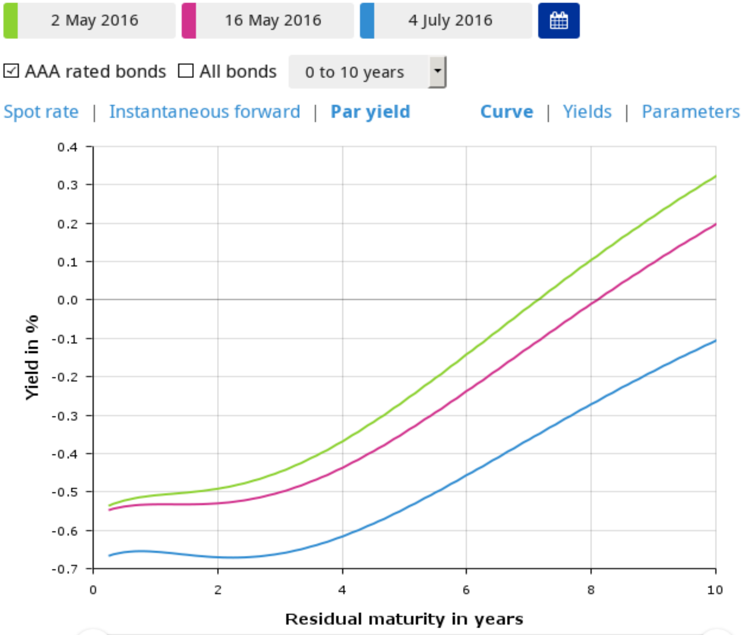
$L(t,T)$: default-free forward Libor rate for the interval T to $T+\delta$

$$L(t,T) := \frac{1}{\delta}\left(\frac{B(t,T)}{B(t,T+\delta)} - 1\right)$$

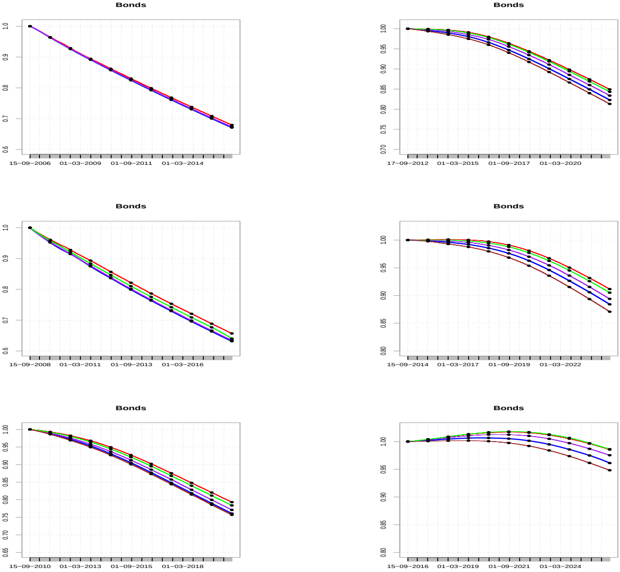
$F_B(t,T,U)$: forward price process for the two maturities T and U

$$F_B(t,T,U) := \frac{B(t,T)}{B(t,U)}$$

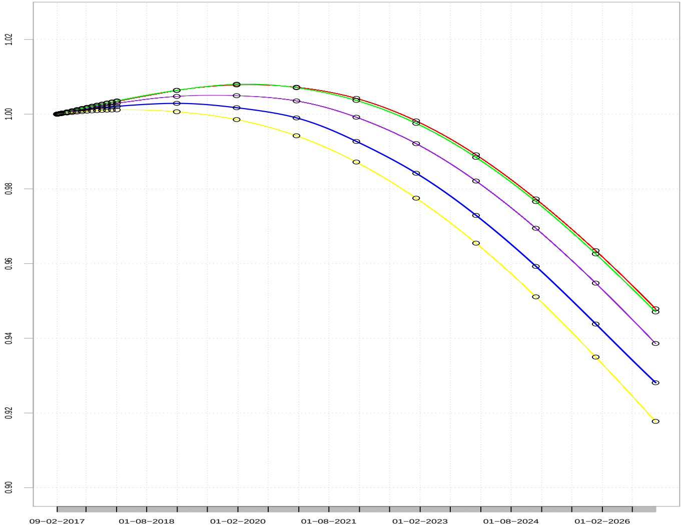
$$\Longrightarrow \quad 1 + \delta L(t,T) = \frac{B(t,T)}{B(t,T+\delta)} = F_B(t,T,T+\delta)$$



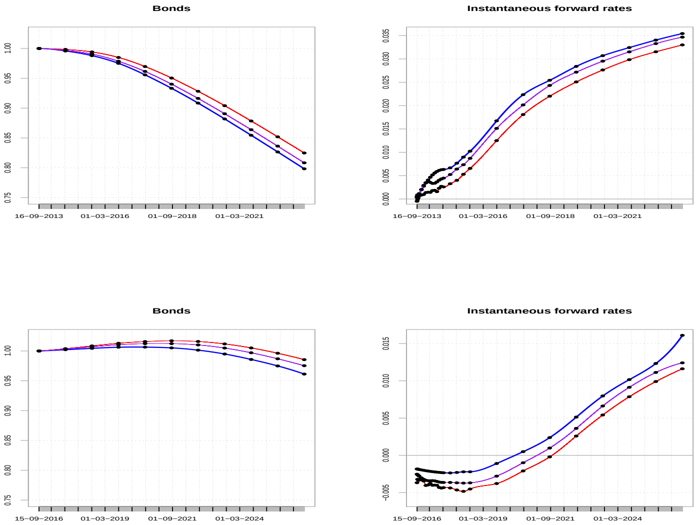
ECB, Frankfurt



Evolution of Discount Curves 2006 – 2016



Discount Curves, February 9, 2017



Derived Instantaneous Forward Rates 2013 and 2016

Bootstrapping of Initial Curves

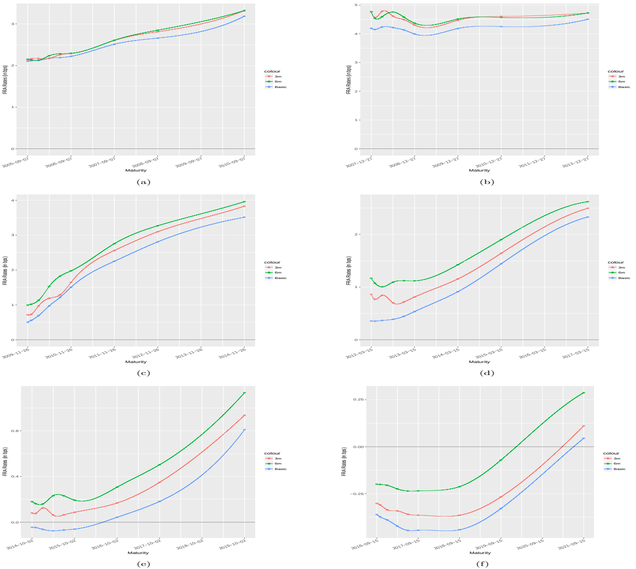
The quoted *overnight indexed swap rate* (OIS) for a discrete tenor structure $\mathcal{T} = \{T_0, \ldots, T_n\}$ with tenor δ can be expressed in the form

$$S_0^{\text{on}}(\mathcal{T}) = \frac{B_0^{\text{d}}(T_0) - B_0^{\text{d}}(T_n)}{\sum_{k=1}^{n} \delta B_0^{\text{d}}(T_k)}$$

Quotes of the swap rates are given for increasing maturities.

From the $B_0^{\text{d}}(T_k)$ we derive for each pair of consecutive dates $T_{k-1}, T_k \in \mathcal{T}$ the *discretely compounded forward reference rates*

$$L^{\text{d}}(0, T_{k-1}, T_k) = \frac{1}{\delta}\left(\frac{B_0^{\text{d}}(T_{k-1})}{B_0^{\text{d}}(T_k)} - 1\right)$$



Historical Evolution of the Tenor-Dependent FRA Curves

The Initial 6-month Curve

Start with its value at the maturity of six months by using the quoted deposit rate $R_0^{6m}(0.5)$

$$B_0^{6m}(0.5) = \frac{1}{1 + 0.5 \cdot R_0^{6m}(0.5)}.$$

For mid and long term maturities from one year upwards proceed by bootstrapping. Use quoted swap rates based on a 6-month floating leg according to

$$S_0(\mathcal{T}^{6m}, \mathcal{T}) = \frac{0.5 \sum_{k=1}^{n_{6m}} B_0^{d}(T_k^{6m}) L^{6m}(0, T_{k-1}^{6m}, T_k^{6m})}{\delta \sum_{l=1}^{n} B_0^{d}(T_l)}$$

where

$$L^{6m}(0, T_{k-1}^{6m}, T_k^{6m}) = \frac{1}{0.5} \left(\frac{B_0^{6m}(T_{k-1}^{6m})}{B_0^{6m}(T_k^{6m})} - 1 \right).$$

Values for the maturities of one and three months are added by using rates of forward rate agreements

Literature on Modelling Multiple Curves

Kijima, Tanaka and Wong (2009)

Kenyon (2010), Mercurio (2009, 2010)

Bianchetti (2010), Morini (2009)

Crépey, Grbac and Nguyen (2012)

Crépey, Douady (2013)

Filipović and Trolle (2013)

Henrard (2010, 2014)

Grbac and Runggaldier (2015)

Cuchiero, Fontana and Gnoatto (2016, 2017)

Grbac, Papapantoleon, Schoenmakers and Skovmand (2015)

Crépey, Macrina, Nguyen and Skovmand (2016)

Eberlein and Gerhart (2018)

Macrina and Mahomed (2018)

Afeus, Grasselli and Schlögl (2018)

The Driving Process

$L = (L^1, \ldots, L^d)$ is a d-dimensional time-inhomogeneous Lévy process, i.e. L has independent increments and the law of L_t is given by the characteristic function

$$\mathbb{E}[\exp(i\langle u, L_t\rangle)] = \exp \int_0^t \theta_s(iu)\, \mathrm{d}s \qquad \text{with}$$

$$\theta_s(z) = \langle z, b_s\rangle + \frac{1}{2}\langle z, c_s z\rangle + \int_{\mathbb{R}^d} \left(e^{\langle z,x\rangle} - 1 - \langle z, x\rangle\right) F_s(\mathrm{d}x)$$

where $b_t \in \mathbb{R}^d$, c_t is a symmetric nonnegative-definite $d \times d$-matrix and F_t is a Lévy measure

Description in Terms of Modern Stochastic Analysis

$L = (L_t)$ is a special semimartingale with canonical representation

$$L_t = \int_0^t b_s \, \mathrm{d}s + \int_0^t c_s^{1/2} \mathrm{d}W_s + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu)(\mathrm{d}s, \mathrm{d}x)$$

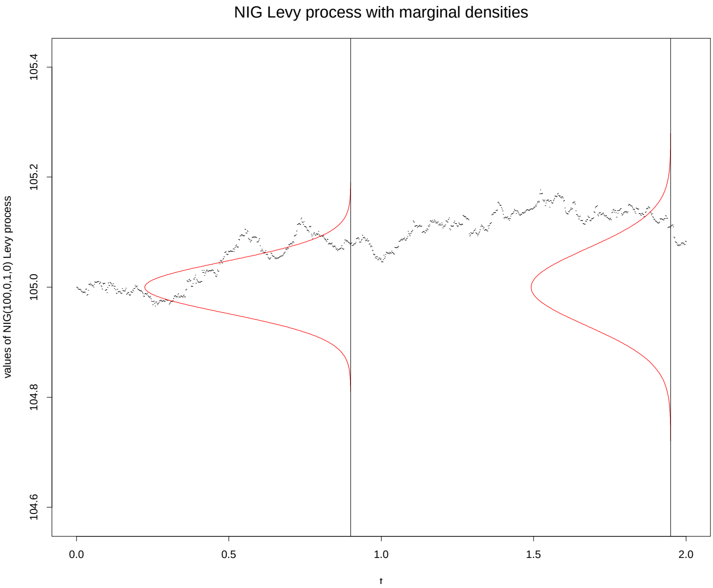
and characteristics

$$A_t = \int_0^t b_s \, \mathrm{d}s, \qquad C_t = \int_0^t c_s \, \mathrm{d}s, \qquad \nu(\mathrm{d}s, \mathrm{d}x) = F_s(\mathrm{d}x) \, \mathrm{d}s$$

$W = (W_t)$ is a standard d-dimensional Brownian motion,

μ^L the random measure of jumps of L and ν is the compensator of μ^L

Simulation of a GH Lévy motion



The Basic Discount Curve (1)

(following Eb., Özkan (2005))

Tenor structure $\mathcal{T}$: $\quad 0 \leq T_0 < T_1 < \cdots < T_n = T^* \quad$ with $T_k - T_{k-1} = \delta$

$B_t^d(T)$: bond price at time t maturing at T

$L^d(t, T_{k-1}, T_k)$: forward reference rate for the interval T_{k-1} to T_k

$$L^d(t, T_{k-1}, T_k) := \frac{1}{\delta}\left(\frac{B_t^d(T_{k-1})}{B_t^d(T_k)} - 1\right)$$

$F^d(t, T_{k-1}, T_k)$: forward price process for the maturities T_{k-1}, T_k

$$F^d(t, T_{k-1}, T_k) := \frac{B_t^d(T_{k-1})}{B_t^d(T_k)}$$

$$\Rightarrow \qquad F^d(t, T_{k-1}, T_k) = 1 + \delta L^d(t, T_{k-1}, T_k)$$

The Basic Discount Curve (2)

Assumptions

($\mathbb{DFP}$.1) The initial term structure of bond prices $B_0^d(T)$ for $T \in [0, T^*]$ is given. This defines the starting values of the forward processes

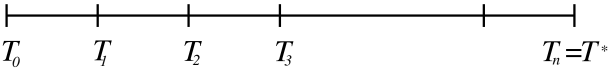
$$F^d(0, T_{k-1}, T_k) = \frac{B_0^d(T_{k-1})}{B_0^d(T_k)}$$

($\mathbb{DFP}$.2) For any maturity $T_{k-1} \in \mathcal{T}$ there is a bounded, continuous and deterministic function

$$\lambda^d(\cdot, T_{k-1}) : [0, T^*] \to \mathbb{R}_+^d.$$

We require that

$$\lambda^d(t, T_{k-1}) = (0, \ldots, 0) \quad \text{for } t > T_{k-1}$$



Backward Induction (1)

We postulate

$$F^d(t, T_{n-1}, T_n) = F^d(0, T_{n-1}, T_n) \exp\left(\int_0^t \lambda^d(s, T_{n-1}) dL_s^{T_n} + \int_0^t b^d(s, T_{n-1}, T_n) ds \right)$$

where $L^{T_n} = L^{T^*}$ is given in the form

$$L_t^{T^*} = \int_0^t \sqrt{c_s}\, dW_s^{T^*} + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu^{T^*})(ds, dx)$$

with $P_{T^*}^d$-Brownian motion W^{T^*} and compensator $\nu^{T^*}(dt, dx)$.

Choose the drift $b^d(\cdot, T_{n-1}, T_n)$ s.t. $F^d(\cdot, T_{n-1}, T_n)$ becomes a $P_{T^*}^d$-martingale

$$b^d(t, T_{n-1}, T_n) = -\frac{1}{2}\lambda^d(t, T_{n-1}) c_t \lambda^d(t, T_{n-1})^\top - \int_{\mathbb{R}^d} (e^{\lambda^d(t, T_{n-1})x} - 1 - \lambda^d(t, T_{n-1})x) F_t^{T^*}(dx)$$

Backward Induction (2)

Define the forward martingale measure associated with T_{n-1}

$$\frac{dP^d_{T_{n-1}}}{dP^d_{T_n}} := \frac{F^d(T_{n-1}, T_n, T_n)}{F^d(0, T_{n-1}, T_n)}$$

By Girsanov's theorem

$$W_t^{T_{n-1}} := W_t^{T^*} - \int_0^t \sqrt{c_s}\, \lambda^d(s, T_{n-1})^\top ds$$

is a $P^d_{T_{n-1}}$-standard Brownian motion and

$$\nu^{T_{n-1}}(dt, dx) := \exp(\lambda^d(t, T_{n-1})x)\nu^{T^*}(dt, dx) = F_t^{T_{n-1}}(dx)dt$$

defines the $P^d_{T_{n-1}}$-compensator of μ^L.

Backward Induction (3)

Proceeding backwards along the tenor structure $\mathcal{T}$ one gets for each $k \in \{1, \dots, n\}$

$$F^d(t, T_{k-1}, T_k) = F^d(0, T_{k-1}, T_k) \exp\left(\int_0^t \lambda^d(s, T_{k-1}) dL_s^{T_k} + \int_0^t b^d(s, T_{k-1}, T_k) ds\right)$$

where

$$L_t^{T_k} = \int_0^t \sqrt{c_s}\, dW_s^{T_k} + \int_0^t \int_{\mathbb{R}^d} x(\mu^L - \nu^{T_k})(ds, dx)$$

and the drift term is chosen s.t. $F^d(\cdot, T_{k-1}, T_k)$ is a $P_{T_k}^d$-martingale.

Multiple Term Structure Curves

Consider m tenor structures nested in $\mathcal{T}$

$$\mathcal{T}^m \subset \cdots \subset \mathcal{T}^1 \subset \mathcal{T}$$
$$\mathcal{T}^i = \{T_0^i, \dots, T_{n_i}^i\}$$
$$0 \leq T_0^i = T_0 < T_1^i < \cdots < T_{n_i}^i = T_n = T^*$$

year fractions: $\delta^i = \delta^i(T_{k-1}^i, T_k^i)$ independent of k

$L^i(T_{k-1}^i, T_k^i)$: T_{k-1}^i-spot Libor/Euribor rate

Define

$$L^i(t, T_{k-1}^i, T_k^i) := \mathbb{E}^d_{T_k^i}[L^i(T_{k-1}^i, T_k^i) \mid \mathcal{F}_t]$$

as the forward reference rate corresponding to δ^i.

Possibilities to Model the Spreads

(1) Additive forward spreads

$$s^i(t, T^i_{k-1}, T^i_k) := L^i(t, T^i_{k-1}, T^i_k) - L^d(t, T^i_{k-1}, T^i_k)$$

or

(2) Multiplicative forward spreads

$$S^i(t, T^i_{k-1}, T^i_k) := \frac{1 + \delta^i L^i(t, T^i_{k-1}, T^i_k)}{1 + \delta^i L^d(t, T^i_{k-1}, T^i_k)} = \frac{1 + \delta^i L^i(t, T^i_{k-1}, T^i_k)}{F^d(t, T^i_{k-1}, T^i_k)}$$

In the forward price framework the natural choice are multiplicative spreads.

Lemma

$$\begin{aligned} & L^i(\cdot, T^i_{k-1}, T^i_k) \quad \text{is a } P^d_{T^i_k}\text{-martingale} \\ \Leftrightarrow \quad & S^i(\cdot, T^i_{k-1}, T^i_k) \quad \text{is a } P^d_{T^i_{k-1}}\text{-martingale} \end{aligned}$$

Model Variant (a)

Choose bounded, continuous, deterministic volatilities $\gamma^i(\cdot, T^i_{k-1})$ and postulate for each pair $T^i_{k-1}, T^i_k \in \mathcal{T}^i$

$$S^i(t, T^i_{k-1}, T^i_k) = S^i(0, T^i_{k-1}, T^i_k) \exp\left(\int_0^t \gamma^i(s, T^i_{k-1}) dL_s^{T^i_{k-1}} + \int_0^t b^i(s, T^i_{k-1}) ds\right)$$

where $b^i(\cdot, T^i_{k-1})$ is chosen s.t. $S^i(\cdot, T^i_{k-1}, T^i_k)$ is a $P^d_{T^i_{k-1}}$-martingale

$\rightarrow$ maximum of tractability

basic forward reference rate as well as the δ^i- forward rates can become negative (the initial rates can be negative)

multiplicative spread not necessarily ≥ 1

Model Variant (a) Continued

We get the following explicit form for the δ^i-forward rate

$$
\begin{aligned}
& 1 + \delta^i L^i(t, T^i_{k-1}, T^i_k) = S^i(t, T^i_{k-1}, T^i_k) F^{\mathrm{d}}(t, T^i_{k-1}, T^i_k) \\
= & \left(1 + \delta^i L^i(0, T^i_{k-1}, T^i_k)\right) \exp\Bigg(\int_0^t \big[\sum_{j \in \mathcal{J}^i_k} \lambda^{\mathrm{d}}(s, T_{j-1}) + \gamma^i(s, T^i_{k-1}) \big] dL_s^{T^i_k} \\
& + \int_0^t \Big[b^i(s, T^i_{k-1}) + \langle \gamma^i(s, T^i_{k-1}), w(s, T^i_{k-1}, T^i_k) \rangle \\
& + \sum_{j \in \mathcal{J}^i_k} \big[\langle \lambda^{\mathrm{d}}(s, T_{j-1}), w(s, T_j, T^i_k) \rangle + b^{\mathrm{d}}(s, T_{j-1}, T_j) \big] \Big] ds \Bigg).
\end{aligned}
$$

Model Variant (b)

Choose again volatilities $\overline{\gamma}^i(\cdot, T^i_{k-1})$ as before and postulate for each pair $T^i_{k-1}, T^i_k \in \mathcal{T}^i$

$$\frac{S^i(t, T^i_{k-1}, T^i_k) - 1}{S^i(0, T^i_{k-1}, T^i_k) - 1} = \exp\Big(\int_0^t \overline{\gamma}^i(s, T^i_{k-1}) dL_s^{T^i_{k-1}} + \int_0^t \overline{b}^i(s, T^i_{k-1}) ds \Big)$$

where

$$\overline{b}^i(t, T^i_{k-1}) = -\frac{1}{2}\overline{\gamma}^i(t, T^i_{k-1}) c_t \overline{\gamma}^i(t, T^i_{k-1})^\top - \int_{\mathbb{R}^d} (e^{\overline{\gamma}^i(t, T^i_{k-1})x} - 1 - \overline{\gamma}^i(t, T^i_{k-1})x) F_t^{T^i_{k-1}}(dx).$$

$\rightarrow$ reasonable tractability

multiplicative spread is > 1

Calibration

Pricing formula for caps with tenor length δ^l: $\quad T + \delta^l = T_k$

$$\begin{aligned}
&\mathrm{Cpl}(t, T, \delta^l, K) \\
&\quad = \delta^l B_t^d(T_k)\mathbb{E}_{T_k}^d\left[\left(L^l(T, T_k) - K\right)^+ \mid \mathcal{F}_t\right] \\
&\quad = B_t^d(T_k)\mathbb{E}_{T_k}^d\left[\left(1 + \delta^l L^l(T, T, T_k) - (1 + \delta^l K)\right)^+ \mid \mathcal{F}_t\right] \\
&\quad = B_t^d(T_k)\mathbb{E}_{T_k}^d\left[\left(F^d(T, T, T_k)S^l(T, T, T_k) - \tilde{K}^l\right)^+ \mid \mathcal{F}_t\right] \\
&\quad = B_t^d(T_k)(Z_t^k)^{-1}\mathbb{E}_{T*}^d\left[Z_T^k\left(F^d(T, T, T_k)S^l(T, T, T_k) - \tilde{K}^l\right)^+ \mid \mathcal{F}_t\right]
\end{aligned}$$

Calibration Continued

Assumption: volatility functions are decomposable

$$\lambda^d(t,T) = \lambda_1^d(T)\lambda(t)$$

and

$$\overline{\gamma}^l(t,T) = \overline{\gamma}_1^l(T)\lambda(t)$$

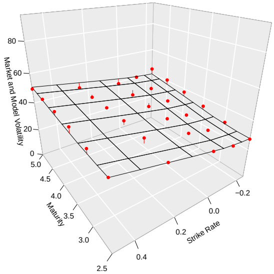
Define $\quad X_T := \int_0^T \lambda(s)\, dL_s^{T^*}$

$\Rightarrow \quad \mathrm{Cpl}(0,T,\delta^l,K) = B_0^d(T_k)\, \mathbb{E}_{T^*}^d[f_K^{k,l}(X_T)] \quad$ for a function $f_K^{k,l}$

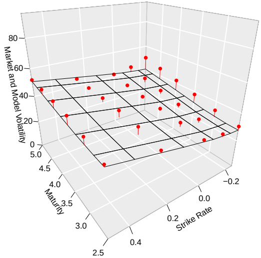
Applying the Fourier-based approach one gets

$$\mathrm{Cpl}(0,T,\delta^l,K) = \frac{B_0^d(T_k)}{\pi} \int_0^\infty \mathrm{Re}\big(\varphi_{X_T}(u - iR)\, \widehat{f}_K^{k,l}(iR - u)\big)\, du$$

Calibration Results



(a) Model (a)



(b) Model (b)

Figure: Calibrated Volatility Surfaces on September 15, 2016

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