Staying at the Zero Lower Bound with Embedded Markov Chain
(Work in Progress)

Christian Gouriéroux (Toronto/TSE) and Yang Lu (Paris 13)

2019 Paris Risk Forum

Introduction

Motivation

- In the past decades affine processes have become the workhorse for interest rate models due to their positivity and tractability for the term structure of yields.
- However they usually do not allow for the short rate to stay at the ZLB [except Monfort et al. (2017)].
- The alternative shadow rate models are not tractable for pricing.

What we do

- This paper proposes models compatible with the ZLB.
- The model is non affine, but is tractable for derivative pricing.
- The modelling is based on an **endogenous** Markov chain.
- The model is quite flexible, i.e. the Markov chain has different regimes for the ZLB and non-ZLB states.

The model

Denote by

- Factors $X_t = (r_t, Y_t)$, where r_t is the short rate, Y_t is the factor(s) driving longer term rates.
- Regimes $Z_t = (\mathbb{1}_{r_t>0}, S_t)$, where S_t is latent, and can take S different values. In other words there are in total $2S$ regimes.

They are defined alternately:

$$\begin{pmatrix} r_{t-1} \\ Y_{t-1} \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{1}_{r_t>0} \\ S_t \end{pmatrix} \longrightarrow \begin{pmatrix} r_t \\ Y_t \end{pmatrix} \longrightarrow \begin{pmatrix} \mathbb{1}_{r_{t+1}>0} \\ S_{t+1} \end{pmatrix} \longrightarrow \begin{pmatrix} r_{t+1} \\ Y_{t+1} \end{pmatrix}.$$

In particular, if $\mathbb{1}_{r_t>0} = 0$, then r_t is zero, that is the ZLB.

We also assume that each variable depends only on its nearest left neighbor.

The conditional distribution

The conditional distributions are characterized by:

- The conditional density of $X_t = (r_t, Y_t)$ given $Z_t = (\mathbb{1}_{r_t>0}, S_t)$:

$$\alpha_{j,s}(x_t) = \alpha_{j,s}(r_t, y_t), \quad j \in \{0,1\}, s \in \{1,...,S\},$$

  or stacked in a vector: $\alpha(x_t) = \begin{pmatrix} \alpha_0(x_t) \\ \alpha_1(x_t) \end{pmatrix}$.

- The vector of conditional probabilities of Z_t given X_{t-1}:

$$\beta(x_{t-1}) = \begin{pmatrix} \beta_0(x_{t-1}) \\ \beta_1(x_{t-1}) \end{pmatrix},$$

  where $\beta_0(x_{t-1}) \in \mathbb{R}^S$, sums up to $\mathbb{P}[r_t = 0 | X_{t-1}]$.

We can show that both (X_t) and (Z_t) are Markov, and (Z_t) is called the Embedded Markov chain (EMC).

The transition distribution of (X_t) resembles that of a standard Markov chain:

Proposition

$$f(r_{t+1}, y_{t+1} | r_t, y_t) = \beta'(r_t, y_t)\alpha(r_{t+1}, y_{t+1}),$$
$$f(r_{t+h}, y_{t+h} | r_t, y_t) = \beta'(r_t, y_t)\Pi^{h-1}\alpha(r_{t+h}, y_{t+h}), \quad \forall h \geq 1,$$

where $\Pi = \int \alpha(r, y)\beta'(r, y)\mathrm{d}\mu(r, y)$

$$= \begin{bmatrix} \int \alpha_0(0, y)\beta_0'(0, y)\mathrm{d}y & \int \alpha_0(0, y)\beta_1'(0, y)\mathrm{d}y \\ \int \alpha_1(r, y)\beta_0'(r, y)\mathrm{d}\mu(r, y) & \int \alpha_1(r, y)\beta_1'(r, y)\mathrm{d}\mu(r, y) \end{bmatrix}$$

$$:= \begin{bmatrix} \Pi_{00} & \Pi_{01} \\ \Pi_{10} & \Pi_{11} \end{bmatrix} \in \mathcal{M}_{2S}(\mathbb{R}),$$

is the transition matrix of the Markov chain (Z_t).

Duration analysis

We want to answer:

- If the economy is at the ZLB ($r_t = 0$), when will we leave?
- If $r_t > 0$, when will we entering into ZLB?
- Moreover, how do these predictors depend on the current term structure?

Closed form survival probabilities

We have:

Proposition

For each horizon h,

$$\begin{aligned}
\mathcal{S}_{00}(h, y_t) &= \mathbb{P}[r_{t+h} = \cdots = r_{t+1} = 0 | \mathbb{1}_{r_t=0} = 1, y_t] \\
&= \beta_0'(0, y_t)\Pi_{00}^{h-1}\mathbb{1}_S \\
\mathcal{S}_{11}(h, r_t, y_t) &= \mathbb{P}[r_{t+h} > 0, \cdots, r_{t+1} > 0 | \mathbb{1}_{r_t=0} = 0, r_t, y_t] \\
&= \beta_1'(r_t, y_t)\Pi_{11}^{h-1}\mathbb{1}_S,
\end{aligned}$$

In particular the two probabilities depend on different block matrices Π_{00} and Π_{11}.

Risk neutral dynamics

- Let us specify the $\mathbb{Q}$-dynamics via the stochastic discount factor (SDF).
- Remind that in the affine framework, under an exponential affine change of measure, $\mathbb{Q}$-dynamics is still affine.
- We will see that similarly for EMC models, the $\mathbb{Q}$-dynamics is still EMC.

SDF

The SDF m_{t+1} between dates t and $t+1$ should satisfy:

$$\mathbb{E}_t^{\mathbb{P}}[m_{t+1}] := \mathbb{E}_t[m_{t+1}] = \exp(-r_t)$$

One multiplicative specification compatible with this constraint is:

$$m_{t+1} = \frac{\exp(-r_t)\kappa(r_{t+1}, y_{t+1})}{\mathbb{E}_t[\kappa(r_{t+1}, y_{t+1})]} = \frac{\exp(-r_t)\kappa(r_{t+1}, y_{t+1})}{\beta'(r_t, y_t) \int \kappa\alpha},$$

where $\kappa(\cdot, \cdot)$ is any positive function.

Risk neutral dynamics

Proposition

The $\mathbb{Q}-$dynamics is still Markov with EMC:

$$f^*(r_{t+1}, y_{t+1}|r_t, y_t) = [\beta^*(r_t, y_t)]'\alpha^*(r_{t+1}, y_{t+1}),$$

with:

$$\alpha^*_{j,s}(r_{t+1}, y_{t+1}) = \frac{\kappa(r_{t+1}, y_{t+1})\alpha_{j,s}(r_{t+1}, y_{t+1})}{\int \kappa\alpha_{j,s}},$$

$$\beta^*_{j,s}(r_t, y_t) = \frac{\beta_{i,s}(r_t, y_t)\int \kappa\alpha_{j,s}}{\beta'(r_t, y_t)\int \kappa\alpha}, \qquad \forall j \in \{0,1\}, s \in \{1, ..., S\}.$$

In particular, "insurance" with payoff $\mathbb{1}_{r_{t+1}=r_{t+2}=\cdots=r_{t+h}=0}$ or $\mathbb{1}_{r_{t+1}>0,\cdots,r_{t+h}>0}$ can be priced in closed form.

Bond pricing

Proposition

The zero-coupon bond price is:

$$B(t,h) = \mathbb{E}[m_{t+1}\cdots m_{t+h}|r_t, y_t] = \frac{e^{-r_t}\beta'(r_t,y_t)}{\beta'(r_t,y_t)\int\kappa\alpha} M_1^{h-1}\int\kappa\alpha$$

where the $(2S \times 2S)$ *matrix* M_1 *is given by:*

$$M_1 = \int e^{-r}\frac{\kappa(r,y)\alpha(r,y)\beta'(r,y)}{\beta'(r,y)\int\kappa\alpha}\mathrm{d}\mu(r,y)$$

- Thus computing the term structure for t, h varying amounts to computing M_1.
- We can show that its largest eigenvalue $\rho < 1$, and the long-run interest rate is $-\log\rho$, which is positive.

Bond pricing

Proposition

The zero-coupon bond price is:

$$B(t,h) = \mathbb{E}[m_{t+1}\cdots m_{t+h}|r_t, y_t] = \frac{e^{-r_t}\beta'(r_t,y_t)}{\beta'(r_t,y_t)\int \kappa\alpha} M_1^{h-1} \int \kappa\alpha$$

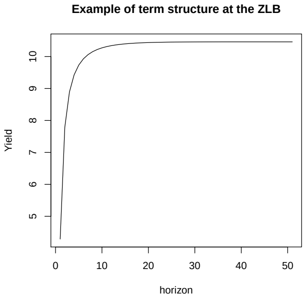
where the $(2S \times 2S)$ *matrix* M_1 *is given by:*

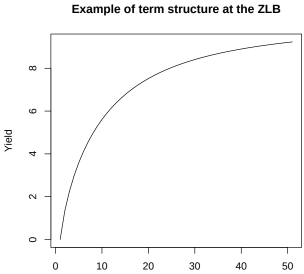
$$M_1 = \int e^{-r} \frac{\kappa(r,y)\alpha(r,y)\beta'(r,y)}{\beta'(r,y)\int \kappa\alpha} \mathrm{d}\mu(r,y)$$

- Thus computing the term structure for t, h varying amounts to computing M_1.
- We can show that its largest eigenvalue $\rho < 1$, and the long-run interest rate is $-\log\rho$, which is positive.

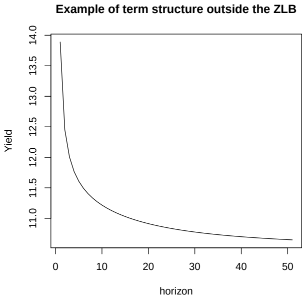
Some illustrations

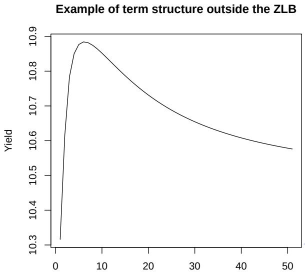
Term structure at the ZLB, with $S = 3$





Term structure outside the ZLB





Conclusion

We have proposed an alternative to the affine term structure models that inherits the tractability of the latter, but is

- compatible with the ZLB
- flexible to distinguish the ZLB and non-ZLB state.
- Next step: estimate the model using bond prices.

$\mathbb{Q}$ dynamics

The risk-neutral conditional density of process (r_t, y_t) is:

$$f^*(r_{t+1}, y_{t+1}|r_t, y_t) = \frac{m_{t+1} f(r_{t+1}, y_{t+1}|r_t, y_t)}{\int m_{t+1} f(r_{t+1}, y_{t+1}|r_t, y_t) \mathrm{d}\mu(r_{t+1}, y_{t+1})}$$
$$= \frac{\kappa(r_{t+1}, y_{t+1})\beta'(r_t, y_t)\alpha(r_{t+1}, y_{t+1})}{\beta'(r_t, y_t) \int \kappa\alpha}.$$