Cross Sectional Momentum

Oh Kang Kwon[†] • Stephen Satchell[†,⋆] (ses999gb@yahoo.com)

Paris – March 2019

[†] Discipline of Finance, The University of Sydney
[⋆] Trinity College, University of Cambridge

Outline

- General thoughts
- Distribution of momentum returns

General Thoughts

What is Cross Sectional Momentum (CSM)?

- We have a universe of assets
- We rank their returns/earnings/prices over a ranking/formation period.
- We go long assets in top m-tile, short bottom m-tile, weighted equally or otherwise
- We then hold our position over a subsequent holding period
- Resulting return on this position is cross sectional momentum (CSM) return
- There are many variations on this structure

Profitability of Momentum

- Some observations:
  1. Momentum is profitable if returns exhibit strong deterministic trend
  2. Momentum is profitable if returns have some autocorrelation
  3. High risk positions sometimes have higher returns
  4. Observation 3 compatible with market efficiency

Momentum – Quantitative Finance

- We argue that cross sectional momentum (CSM) is profitable when there are large differences in expected returns (high factor cross sectional volatility (CSV))
  1. Europe/Asia should be good for CSM (different countries and industries)
  2. UK/US should be bad for CSM (homogeneous) but UK good for momentum?
  3. Japan?

Momentum – BF/Psychology

- Practitioners of behavioural finance would say, implicitly, Brits/Europeans and would have persistent psychological problems that do not correct
- Americans do not have these problems
- Quantitative explanation seems more plausible

Momentum – Quantitative and Behavioural

- When quantitative finance became ugly (2007–2008), it re-emerged as behavioural finance
- Academics were hired to tell tales about investors' incurable psychological issues
- For example, Hong and Stein (1999) with different trader types under-reaction to overconfidence and overreaction to biased self-attribution
- For a prospect theoretical interpretation of momentum returns, see Menkhoff and Schmeling (2006)

Distribution of CSM Returns

- Based on 2017 JEDC paper by Oh Kang Kwon and Stephen Satchell
- Considers the CSM returns as two-period problem – ranking period and holding period
- Assumes the stock returns over the two periods are multivariate normal
- If two periods are independent and returns are stationary, then markets are efficient and high momentum returns are a consequence, presumably, of higher risk
- Construct portfolios consisting of m long and m short assets from a universe of n assets – more generally m_+ long and m_- short

- Consider the special case $n = 2$ and $m = 1$, and let $r_{1,t}$ and $r_{2,t}$ be the returns on two assets over the ranking period, and $r_{1,t+1}$ and $r_{2,t+1}$ the corresponding returns over the holding period
- Then for CSM strategy:
  - if $r_{1,t} > r_{2,t}$, viz. in ranking period, then long asset 1 and short asset 2
  - if $r_{1,t} < r_{2,t}$, then do the opposite
- This implies for resulting CSM return, $r_{\text{csm},t+1}$, over holding period

$$\begin{aligned}\text{pdf}(r_{\text{csm},t+1}) &= \text{pdf}(r_{1,t+1} - r_{2,t+1} \mid r_{1,t} > r_{2,t}) \\ &\quad + \text{pdf}(r_{2,t+1} - r_{1,t+1} \mid r_{1,t} < r_{2,t})\end{aligned}$$

- If markets are efficient, this is a mixture of univariate normals

$$\begin{aligned}\text{pdf}(r_{\text{csm},t+1}) &= \text{pdf}(r_{1,t+1} - r_{2,t+1})\text{prob}(r_{1,t} > r_{2,t}) \\ &\quad + \text{pdf}(r_{2,t+1} - r_{1,t+1})\text{prob}(r_{1,t} < r_{2,t}),\end{aligned}$$

  in this case, kurtotic and skewed for plausible parameter values

- If markets are not efficient (predictable), then structure is more complicated but given in terms of truncated normals

$$
\begin{aligned}
&\text{pdf}(r_{\text{csm},t+1}) \\
&\quad = \phi_1\left(r; -\mu_{t+1}, \varsigma_{t+1}^2\right)\Phi_1\left[0; \mu_t - \frac{\varrho_{t,t+1}\varsigma_t}{\varsigma_{t+1}}\left(r+\mu_{t+1}\right), \varsigma_t^2\left(1-\varrho_{t,t+1}^2\right)\right] \\
&\quad + \phi_1\left(r; \mu_{t+1}, \varsigma_{t+1}^2\right)\Phi_1\left[0; -\mu_t - \frac{\varrho_{t,t+1}\varsigma_t}{\varsigma_{t+1}}\left(r-\mu_{t+1}\right), \varsigma_t^2\left(1-\varrho_{t,t+1}^2\right)\right],
\end{aligned}
$$

where for $i, j \in \{1,2\}$ and $u \in \{t, t+1\}$

$$
\mu_{t+1} = \mathbb{E}[r_{2,t+1} - r_{1,t+1}], \qquad \sigma_{i,u}^2 = \text{var}(r_{i,u}), \quad \rho_u = \frac{\text{cov}(r_{1,u}, r_{2,u})}{\sigma_{1,u}\sigma_{2,u}},
$$

$$
\rho_{i,j} = \frac{\text{cov}(r_{i,t}, r_{j,t+1})}{\sigma_{i,t}\sigma_{j,t+1}}, \quad \varsigma_t^2 = \sigma_{1,u}^2 + \sigma_{2,u}^2 - 2\rho_u\sigma_{1,u}\sigma_{2,u},
$$

$$
\begin{aligned}
\varsigma_{t,t+1} &= \rho_{1,1}\sigma_{1,t}\sigma_{1,t+1} + \rho_{2,2}\sigma_{2,t}\sigma_{2,t+1} - \rho_{1,2}\sigma_{1,t}\sigma_{2,t+1} - \rho_{2,1}\sigma_{2,t}\sigma_{1,t+1}, \\
\varrho_{t,t+1} &= \frac{\varsigma_{t,t+1}}{\varsigma_t\varsigma_{t+1}}
\end{aligned}
$$

- Analytic expressions for first four central moments of $r_{\text{csm},t+1}$ available

- When are momentum returns positive?
- Consider again the simple case $n = 2$, $m = 1$, and market efficient
- Letting $p = \text{prob}(r_{1,t} > r_{2,t})$,

$$\begin{aligned}\mathbb{E}[r_{\text{csm},t+1}] &= p\left(\mathbb{E}[r_{1,t+1}] - \mathbb{E}[r_{2,t+1}]\right) \\ &\quad + (1-p)\left(\mathbb{E}[r_{2,t+1}] - \mathbb{E}[r_{1,t+1}]\right) \\ &= (2p-1)\left(\mu_{1,t+1} - \mu_{2,t+1}\right),\end{aligned}$$

where $\mu_{i,t+1} = \mathbb{E}[r_{i,t+1}]$ and

$$p = \Phi\left(\frac{\mu_{1,t} - \mu_{2,t}}{\sqrt{\sigma_{1,t}^2 + \sigma_{2,t}^2 - 2\rho_t \sigma_{1,t}\sigma_{2,t}}}\right)$$

- So you have to be able to pick the stock with the higher expected return more than 50% of the time – not surprising!

Distribution of CSM Returns and CSV

- Would high cross sectional volatilities (CSV) be good/bad for CSM?
- This depends on whether it is factor CSV (good) or idiosyncratic CSV (bad)
- We can see this from previous formula, factor CSV increases the numerator while idiosyncratic CSV the denominator

Special Case $n = 3$ and $m = 1$

- Ordering of asset returns in the ranking period corresponds to $6 = 3!$ permutations of $\{1, 2, 3\}$, viz. $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$
- If $(1, 2, 3)$, we long asset 1 and short asset 3, if $(1, 3, 2)$ we long asset 1 and short asset 2, etc.
- For a permutation π of $\{1, 2, 3\}$, write π_i for the image of i so that, for example, if $\pi = (2, 1, 3)$ then $\pi_1 = 2$, $\pi_2 = 1$ and $\pi_3 = 3$. Then

$$\text{pdf}(r_{\text{csm},t+1}) = \sum_{\pi \in S_3} \text{pdf}(r_{\pi_1,t+1} - r_{\pi_3,t+1} \mid r_{\pi_1,t} > r_{\pi_2,t} > r_{\pi_3,t}),$$

  where π ranges over all permutations S_3 of $\{1, 2, 3\}$
- Resulting distribution is from the unified skew-normal (SUN) family considered in Arellano-Valle and Azzalini (2006)

General Case of n Assets

- Above results generalize naturally to universe of n assets, m_+ long and m_- short:
  - pdf for CSM return is a sum over the permutations of $\{1, 2, \ldots, n\}$
  - each term in the CSM return pdf consists of univariate normal and truncated multivariate normal
  - total number of distinct orderings of asset returns over ranking period is

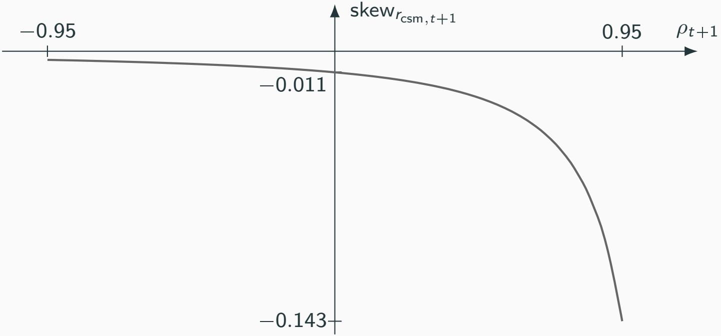
$$\frac{n!}{(n - m_+ - m_-)!m_+!m_-!}$$

  - if we want to investigate S&P500 long top 100 and short bottom 100, this number is vast

- So is this profoundly useless?
  - Perhaps for direct practical applications
  - We at least understand why momentum returns should be kurtotic
  - Even with normal returns we get non-normal momentum returns
  - Too early to link volatility spikes with momentum crashes, but framework may be able to address this – $r_{csm,t+1}$ skewness as function of correlation
  - fails to explain why long CSM makes most of the money

Link between Skewness and Correlation

- Special case where return correlations over holding period are all ρ_{t+1}



- S&P500 and Fama-French data suggests skewness of CSM returns tend to be negative
- $\rho_{t+1} \to 1$ related to market crashes as Sancetta and Satchell (2007) show that $\rho_{t+1} \to 1$ in a CAPM framework when market vol goes up

Complexity

- Notion that expected utility maximisers take expected values over such a distribution becomes fanciful without access to modern MC
- For a long 50 and short 50 momentum portfolio from S&P500, distinct orderings over the ranking period is

$$\frac{500!}{400!50!50!} \approx 10^{160}$$

which is huge!

- To put things into perspective, number of seconds in the history of the universe is approximately 10^{20}

Conclusion

- We have derived the pdf of CSM returns
- This pdf is recognisable as a density from a known family of distributions
- Results are practically usable only for small n
- For $n = 2$, we can capture many of the stylised facts of CSM returns