Branching Processes for Multiple Curve Modeling

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- Since the 2007-2008 crisis, the **credit** and **funding/liquidity risks** implicit in interbank transactions have deeply affected fixed income markets:
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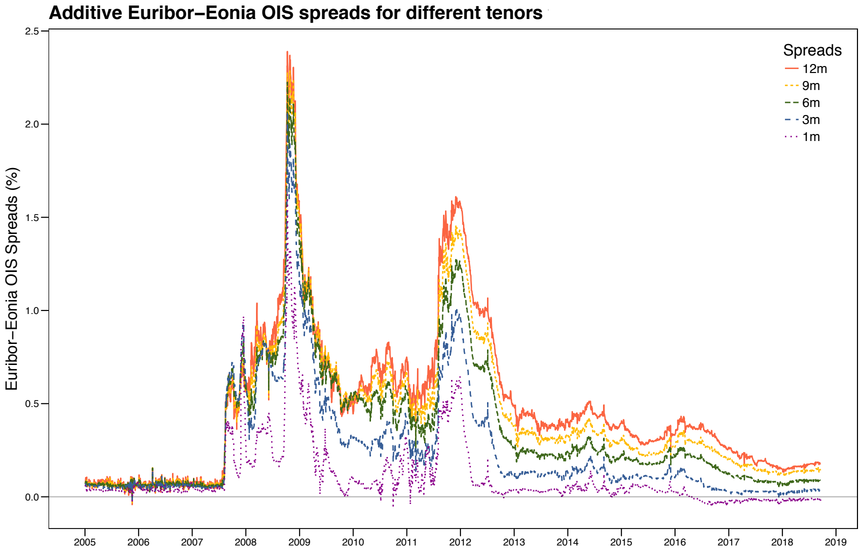
- In recent years, also as a consequence of expansionary monetary policies, interest rates have been persistently low (and even negative).
- Consequences for financial modeling:
  - interbank (Ibor) rates are risky;
  - classical no-arbitrage relations do not hold.
  - persistently low/negative interest rates.

Multiple interest rate curves, where each interest rate (yield) curve is constructed from products depending on a specific tenor (1W, 1M, 3M, 6M, 1Y).

$\Rightarrow$ this is reflected by the presence of spreads between Ibor and OIS rates.

Euribor - OIS spreads



Source: ECB.

Multiple curve modeling

Empirical features of spreads

- generally positive;
- longer tenors are associated to higher spreads;
- strong comovements and common upward jumps;
- volatility clustering.

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The modeling approach

Continuous-state branching processes with immigration (CBI processes) to model

- OIS short rate;
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Main properties

- Consistent with empirical features;
- order relations between Ibor rates associated to different tenors;
- analytical tractability and efficient valuation formulae ($\Rightarrow$ calibration);
- automatic fit to the initially observed term structures.

An (incomplete) overview of modelling approaches

- Fundamental approaches:
  Crépey and Douady (2013), Filipović and Trolle (2013), Chang and Schlögl (2014), Gallitschke et al. (2017), Alfeus et al. (2018).
- Short rate models:
  Kijima et al. (2009), Kenyon (2010), Morino and Runggaldier (2014), Grasselli and Miglietta (2016), Grbac et al. (2016), **Cuchiero - F - Gnoatto (2019)**.
- Libor market models and forward models:
  Mercurio (2010,...,2018), Grbac et al. (2015), Papapantoleon and Wardenga (2018), Eberlein et al. (2018).
- HJM models:
  Moreni and Pallavicini (2010), Pallavicini and Tarenghi (2010), Fujii et al. (2010,2011), Crépey et al. (2012,2015), CFG (2016).
- Rational models:
  Nguyen and Seifried (2015), Crépey et al. (2016), Filipović et al. (2017), Macrina and Mahomed (2018).

Textbooks: Bianchetti and Morini (2013), Grbac and Runggaldier (2015).

Precursory works: Jarrow and Turnbull (1996), Douady and Jeanblanc (2002).

Ibor and OIS rates

- $\boldsymbol{L(t,t,\delta)}$: **Ibor rate** at date t for the period $[t, t+\delta]$;
  we consider a finite set $\mathcal{D}$ of tenors $\delta_1 < \ldots < \delta_m$, for $m \in \mathbb{N}$.

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  (proxy of risk-free rate in market practice).

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- **OIS rate**: fair swap rate for an Overnight Indexed Swap
  (proxy of risk-free rate in market practice).
- From OIS rates we can compute
  - the term structure of **OIS zero-coupon bond** prices: $T \mapsto B(t,T)$;
  - simply compounded OIS forward rates

$$L^{\text{OIS}}(t,t,\delta) := \frac{1}{\delta}\left(\frac{1}{B(t,t+\delta)} - 1\right).$$

In the post-crisis market: $L(t,t,\delta) \neq L^{\text{OIS}}(t,t,\delta)$

- We denote by $(r_t)_{t\geq 0}$ the **short rate** associated to OIS zero-coupon bonds.

Multiplicative spreads

Besides r_t, we take **spot multiplicative spreads** as the main modeling quantities:

$$S^{\delta}(t,t) := \frac{1+\delta L(t,t,\delta)}{1+\delta L^{\mathrm{OIS}}(t,t,\delta)}, \qquad \text{for } \delta \in \mathcal{D}.$$

- Directly observable from rates quoted on the market;
- **market expectation at date t of the interbank risk over $[t, t+\delta]$**;
- typical market behavior:
  - $S^{\delta_i}(t,t) \geq 1$, for all $i = 1, \ldots, m$;
  - $S^{\delta_i}(t,t) \leq S^{\delta_j}(t,t)$, for all $i, j = 1, \ldots, m$ such that $\delta_i < \delta_j$.

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Let also define **forward** multiplicative spreads:

$$S^{\delta}(t,T) := \frac{1+\delta L(t,T,\delta)}{1+\delta L^{\mathrm{OIS}}(t,T,\delta)}, \qquad \text{for } \delta \in \mathcal{D} \text{ and } 0 \leq t \leq T,$$

where $L(t,T,\delta)$ is the forward Ibor rate (fair rate of a FRA).

$\Rightarrow$ compare with today's talk by Ernst Eberlein.

A flow of CBI processes

Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ be a filtered probability space supporting:

- a white noise $W(\mathrm{d}s, \mathrm{d}u)$ on $(0, +\infty)^2$ with intensity $\mathrm{d}s\,\mathrm{d}u$;
- a Poisson time-space random measure $M(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ on $(0, +\infty)^3$ with intensity $\mathrm{d}s\,\pi(\mathrm{d}z)\,\mathrm{d}u$ and compensator $\widetilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$.

For each $i = 1, \ldots, m$, let $Y^i = (Y^i_t)_{t \geq 0}$ be the unique strong solution of

$$Y^i_t = y^i_0 + \int_0^t (\beta(i) - bY^i_s)\mathrm{d}s + \sigma \int_0^t \int_0^{Y^i_s} W(\mathrm{d}s, \mathrm{d}u) + \eta \int_0^t \int_0^{+\infty} \int_0^{Y^i_{s-}} z\widetilde{M}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u),$$

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where

- $\beta : \{1, \ldots, m\} \to \mathbb{R}_+$, with $\beta(i) \leq \beta(i+1)$;
- $(b, \sigma, \eta) \in \mathbb{R}^3_+$;
- π is a tempered alpha-stable measure, explicitly given by

$$\pi(\mathrm{d}z) = \frac{1}{\Gamma(-\alpha)\cos(\pi\alpha/2)} \frac{e^{-\theta z}}{z^{1+\alpha}} \mathbf{1}_{\{z>0\}}\mathrm{d}z,$$

for some parameters $\alpha \in (1, 2)$ and $\theta > \eta$.

$\{Y^i; i = 1, \ldots, m\}$ is a **flow of CBI processes** (see Dawson & Li, 2012).

Modeling multiple curves via a flow of CBI processes

Given the flow of CBI processes $Y = \{Y^i; i = 1, \ldots, m\}$, we specify the OIS short rate and spot multiplicative spreads as

$$r_t = \ell(t) + \mu^\top Y_t,$$
$$\log S^{\delta_i}(t,t) = c_i(t) + Y^i_t,$$

for all $t \geq 0$ and $i = 1, \ldots, m$, where $\ell : \mathbb{R}_+ \to \mathbb{R}$ and $c_i : \mathbb{R}_+ \to \mathbb{R}_+$.

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- spreads are by construction greater than one;
- each process Y^i is a self-exciting mean-reverting process;
- the processes $\{Y^i; 1, \dots, m\}$ are driven by the same sources of randomness;
- strong dependence among different spreads and OIS rates;
- spreads have a mutually exciting behavior: a large value of $S^{\delta_i}(t,t)$ increases the likelihood of upward jumps of all spreads with tenor $\delta_j > \delta_i$.

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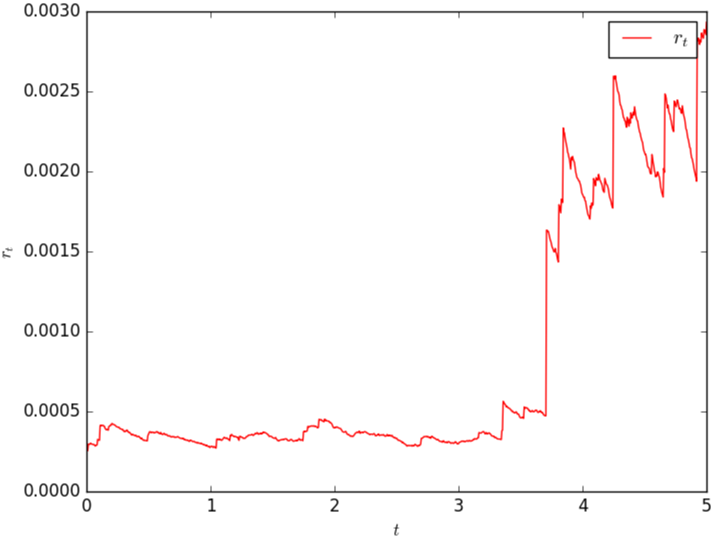
Proposition (monotonicity of spreads)

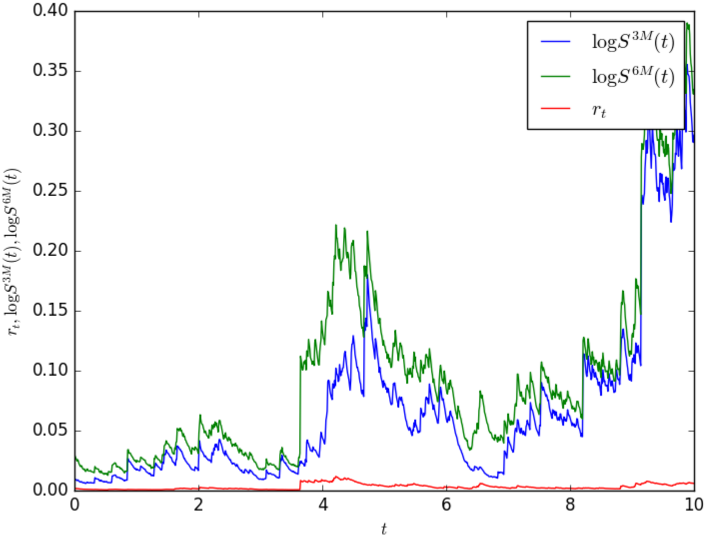
Suppose that $c_i(t) \leq c_{i+1}(t)$ and $y^i_0 \leq y^{i+1}_0$, for all $i = 1, \ldots, m-1$ and $t \geq 0$. Then $S^{\delta_i}(t,T) \leq S^{\delta_{i+1}}(t,T)$ a.s., for all $i = 1, \ldots, m-1$ and $0 \leq t \leq T < +\infty$.

A sample path: OIS rate



Compare also with Jiao et al. (2017), sovereign interest rate modeling.

A sample path: multiplicative spreads



The affine property

CBI processes belong to the class of **affine processes** (Duffie et al., 2003):

$$\mathbb{E}\left[e^{-pY_t^i}\right] = \exp\left(-y_0^i v(t,p) - \beta(i)\int_0^t v(s,p)\,\mathrm{d}s\right), \qquad \text{for all } t \geq 0,$$

where the function $v(\cdot,p)$ is given by the unique solution to the ODE

$$\partial_t v(t,p) = -\phi\big(v(t,p)\big), \qquad v(0,p) = p,$$

with

$$\phi(z) = bz + \frac{\sigma^2}{2}z^2 + \frac{\theta^\alpha + z\alpha\eta\theta^{\alpha-1} - (z\eta+\theta)^\alpha}{\cos(\pi\alpha/2)}, \qquad \text{for } z \geq -\theta/\eta.$$

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By relying on the affine property, we study the following features of the model:

- existence of **exponential moments** of Y^i. In particular,

$$b \geq \frac{\sigma^2}{2}\frac{\theta}{\eta} + \eta\frac{(1-\alpha)\theta^{\alpha-1}}{\cos(\pi\alpha/2)} \qquad \Longrightarrow \qquad \mathbb{E}[e^{Y^i_T}] < +\infty \quad \text{for all } T \geq 0.$$

- **0 is an inaccessible boundary** for Y^i if and only if $\beta(i) \geq \sigma^2/2$;
- characterization of the **ergodic distribution** of the flow.

OIS bond prices and forward multiplicative spreads

The affine property is crucial for **pricing applications**:

1. **OIS zero-coupon bond** prices are given by

$$B(t,T) = \exp\left(\mathcal{A}_0(t,T) + \mathcal{B}_0(T-t)^\top Y_t\right)$$

2. **forward multiplicative spreads** are given by

$$S^{\delta_i}(t,T) = \exp\left(\mathcal{A}_i(t,T) + \mathcal{B}_i(T-t)^\top Y_t\right),$$

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These formulae allow for a direct evaluation of linear interest rate derivatives:

- **Forward Rate Agreements**:

$$\Pi^{\mathrm{FRA}}(t; T, \delta_i, K, N) = N\Big(B(t,T)S^{\delta_i}(t,T) - (1+\delta_i K)B(t,T+\delta_i)\Big);$$

- **Interest Rate Swaps and Basis Swaps**;
- **Convexity adjustments** of the form $\mathbb{E}[L(T,T,\delta_i)|\mathcal{F}_t] - L(t,T,\delta_i)$.

Caplet pricing

Non-linear interest rate derivatives can be efficiently priced by combining

- knowledge of the characteristic function of the CBI flow;
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Consider a **Caplet** with payoff $(L(T,T,\delta_i) - K)^+$ delivered at time $T + \delta_i$:

$$\Pi^{\mathrm{CPL}}(t;T,\delta_i,K) = B(t,T+\delta_i)\mathbb{E}^{T+\delta_i}\left[\left(e^{\mathcal{X}^i_T} - \bar{K}_i\right)^+ \middle| \mathcal{F}_t\right],$$

where $\mathcal{X}^i_T := \log(S^{\delta_i}(T,T)/B(T,T+\delta_i))$ and $\bar{K}_i := 1 + \delta_i K$.

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where $\mathcal{X}_T^i := \log(S^{\delta_i}(T,T)/B(T,T+\delta_i))$ and $\bar{K}_i := 1+\delta_i K$. Let

$$\Phi_{t,T}^i(\zeta) := B(t,T+\delta_i)\mathbb{E}^{T+\delta_i}[e^{\mathrm{i}\zeta\mathcal{X}_T^i}|\mathcal{F}_t]$$

be the modified characteristic function of $\mathcal{X}_T^i$, which can be explicitly computed. Then

$$\Pi^{\mathrm{CPL}}(t;T,\delta_i,K) = R_{t,T}^i(\bar{K}_i) + \frac{1}{\pi}\int_{0-\mathrm{i}\epsilon}^{\infty-\mathrm{i}\epsilon}\mathrm{Re}\left(e^{-\mathrm{i}\zeta\log(\bar{K}_i)}\frac{\Phi_{t,T}^i(\zeta-1)}{-\zeta(\zeta-1)}\right)\mathrm{d}\zeta,$$

where $R_{t,T}^i(\bar{K}_i)$ is a (possibly null) residue term depending on ϵ.

Compare also with Lee (2004), Cuchiero et al. (2019).

Conclusions and outlook

- CBI processes allow to reproduce most of the empirical features of multi-curve spreads in post-crisis interest rate markets:
  - volatility clustering;
  - strong comovements of spreads;
  - persistence of low/negative rates.
- the affine property leads to efficient valuation techniques;
- Work in progress: calibration to market data on caps/floors volatility surface, with two tenors (OIS, 3M and 6M) by FFT and quantization methods.

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Thank you for your attention!