Portfolio Rho-presentativity

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Introduction and Outline

Setup and notations

Setup: Investment universe of n risky assets with covariance matrix $\Sigma \succ 0$.

Π : set of unlevered long-short portfolios (ℓ^1 ball in $\mathbb{R}^n$).

Π^+ : set of unlevered long-only portfolios (standard simplex).

Standard Portfolios:

EW: the Equally-Weighted, $w_{ew} = \mathbf{1}/n$.

EVW: the Equal-Volatility-Weighted, $w_{evw} = \frac{\mathbf{1} \oslash \sigma}{\langle \mathbf{1}, \mathbf{1} \oslash \sigma \rangle}$.

ERC: the Equal Risk Contribution w_{erc} solves in Π^+

$$w_{erc} \odot \Sigma w_{erc} = \frac{\sigma(w_{erc})^2}{n} \mathbf{1} \quad \text{(due to } \sigma(w)^2 = \langle \mathbf{1}, w \odot \Sigma w \rangle\text{)}.$$

MV: the Minimum Variance w_{mv} minimizes $\sigma(w)$ over Π^+.

MDP: the Most Diversified Portfolio w^* maximizes over Π^+ the Diversification Ratio

$$DR(w) := \frac{\langle w, \sigma \rangle}{\sigma(w)} \quad \text{(due to } \sigma(w) \leq \langle w, \sigma \rangle\text{)}.$$

Rho-presentative Portfolios

An Alternative Portfolio Representation

The usual representation through weights has limitations: not holding any financial stock does not necessarily mean no exposure to the financial sector.

Definition: Consider the *correlation spectrum* of a portfolio

$$\rho(w)_i := \varrho(w, e_i)$$

which is the vector of correlations of the portfolio w to *all* assets e_i.

- well-defined for any long-short portfolio $w \neq 0$ as $\Sigma \succ 0$.
- equivalent to weights as $\rho : \Pi \to \mathcal{E} := \{||\cdot||_{C^{-1}} = 1\}$ is bijective.
- measures a normalized and signed exposure of the portfolio w to *each* asset of the universe, or how "well" each asset is represented in the portfolio.

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Rho-presentative Portfolios: Definition and Properties

This alternative representation naturally leads to a new class of portfolios:

Definition

Given an investment universe of risky assets, a portfolio is:

- Representative if $w \succ 0$,
- Rho-presentative if $\rho(w) \succ 0$.

Proposition

- *Representative, not Rho-presentative in general:*

  Cap-weighted indices, EW, EVW.

- *Rho-presentative portfolios:*

  ERC, MV, MDP.

Key Idea: maximize the *overall* exposure of a portfolio to all assets.

Maximally Rho-presentative Portfolios

Definition of Maximally Rho-presentative Portfolios

Definition: an unlevered portfolio $w_f \in \Pi$ is *maximally Rho-presentative* if there is an *aggregation* $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$w_f \in \operatorname*{argmax}_{\Pi} f \circ \rho$$

with f that is

- increasing: if $\rho(w) \succ \rho(y)$ then $f \circ \rho(w) > f \circ \rho(y)$ to advantage portfolios with higher exposures.
- concave: consistent with

$$w_\theta \in]w_0, w_1[\implies \rho(w_\theta) \in d_\theta \times]\rho(w_0), \rho(w_1)[\quad \text{(with } d_\theta > 1\text{)}.$$

- symmetric: aggregated exposure is invariant under permutation.

Existence: as f is concave on $\mathbb{R}^n$, the existence follows from the continuity of $f \circ \rho$ and the fact that ρ is homogeneous of degree 0.

Uniqueness: if f is increasing and concave, the objective of a strict convex combination can always be improved.

Characterization of Maximally Rho-presentative Portfolios

We denote:

- $v^{\uparrow}$ (resp. $v^{\downarrow}$) the sorted v with the max at the top (resp. bottom)
- Volatility adjusted weights $\phi : \Pi^{+} \to \Pi^{+}$ defined by $\phi(w) := \frac{1}{\langle w,\sigma \rangle} w \odot \sigma$.

Theorem

The set $\mathcal{R}$ of unlevered Maximally Rho-presentative portfolios of n assets

$$\mathcal{R} = \left\{ w \in \Pi^{+}, \langle \phi(w)^{\uparrow}, \rho(w)^{\downarrow} \rangle = \langle \phi(w), \rho(w) \rangle \right\}$$

is a finite union of polytopes, with $\phi(\mathcal{R})$ having a small Lebesgue measure

$$\lambda_{n-1}(\phi(\mathcal{R})) \leq \frac{1}{n!} \lambda_{n-1}\left(\Pi^{+}\right).$$

Lemma

For any $y \in \Pi \setminus \Pi^{+}$, there exists $w \in \Pi^{+}$ such that $\rho(w) \succ \rho(y)$.

Other Properties of Maximally Rho-presentative Portfolios

Proposition

A Maximally Rho-presentative w is:

- *weakly Rho-presentative as its average exposure is positive with*

$$n^{-1}\langle \rho(w), \mathbf{1} \rangle \geq DR^{-1}(w) \geq DR(w^*)^{-1} > 0.$$

- *fairly diversified, and positively correlated to a special long-only portfolio:*

$$DR(w) \geq \frac{DR(w_{evw})}{\varrho(w, w_{evw})} > 0.$$

$\Longrightarrow$ EVW w_{evw} and MDP w^* are candidates for being max Rho-presentative.

Applications

A New Framework for Constructing and Comparing Portfolios

Investment Strategy Name	Primal approach: Portfolios maximize $f \circ \rho(w) =$	Dual approach: Weights proportional to	Long Only	ρ-pr	max ρ-pr
EVW	$\langle \rho(w), \mathbf{1} \rangle$	$\mathbf{1} \oslash \sigma$	$\times$		$\times$
ERC	$\langle \ln(\rho(w)), \mathbf{1} \rangle$	$w_i(\Sigma w)_i = \sigma^2(w)/n$	$\times$	$\times$	$\times$
MDP	$\min \rho(w)$	$\text{argmax}_{\Pi^+}\ DR$	$\times$	$\times$	$\times$
Mean-Var ρ	$\mathbb{E}(\rho(w)) - \frac{\lambda}{2}\mathbb{V}ar(\rho(w))$		$\times$		$\lambda \in [0, 1)$

where ρ-pr stands for *Rho-presentative*.

- The EVW, ERC and MDP are maximally Rho-presentative.
- Primal objectives are comparable $\implies$ unified financial interpretation.
- Mean-Var trade off: maximize average vs. dispersion of exposures.

On the Impact of Maximum Weight Constraints

When nearing implementation, practitioners use *maximum weights* $\frac{1}{r}$:

- when imposed by regulators,
- when using objective functions subject to estimation uncertainty.

Theorem

$$\underset{\substack{\langle w,\mathbf{1}\rangle=1\\ 0\leq w_i\leq \frac{1}{r}}}{\text{argmin}}\ \sigma(w) = \underset{\langle w,\mathbf{1}\rangle=1}{\text{argmin}}\ \sigma_{\Sigma_{\lambda,\mu}}(w) = \underset{w\in\Pi}{\text{argmax}}\ \underbrace{\frac{1}{r}\sum_{i=1}^{r}\left[(\rho(w)\odot\sigma)^{\downarrow}\right]_i}_{\textit{a “robust”}\ \min(\rho(w)\odot\sigma)}.$$

- identifies exactly the impact of maximum weights on the objective.
- extends a result by Jagannathan and Ma (2003), that provided an interpretation of the second minimization as using a robust objective.

Investment Strategy Name	Primal approach: Portfolios maximize $f\circ\rho(w) =$	Dual approach: Weights proportional to	Long Only	ρ-pr	max ρ-pr
Constrained MV	$\sum_{i=1}^{r}\left[(\rho(w)\odot\sigma)^{\downarrow}\right]_i$	$\text{argmin}_{\Pi^{+}_{\mathbf{1},r}}\ \sigma_\Sigma$	×		
Constrained MDP	$\sum_{i=1}^{r}\left[(\rho(w))^{\downarrow}\right]_i$	$\text{argmax}_{\Pi^{+}_{\sigma,r}}\ DR$	×		×

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Realized Maximally Rho-presentativity

Goal: Identify Maximally Rho-presentative funds from their observed returns only.

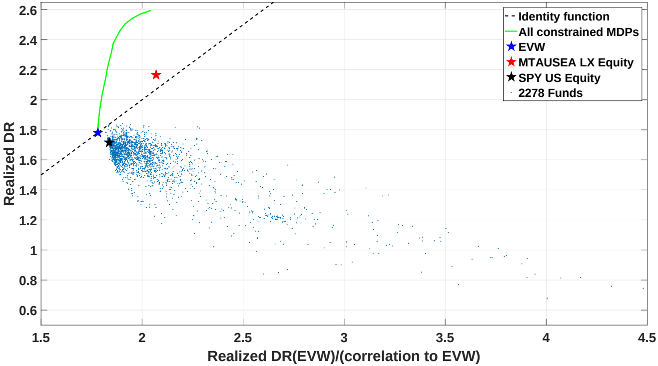
- Funds that are max Rho-presentative necessarily satisfy:

$$DR(w) \geq DR(w_{evw})/\varrho(w, w_{evw}).$$

- Problem: portfolio composition needed for evaluating $DR(w) = \frac{\langle w, \sigma \rangle}{\sigma(w)}$.
- Solution: Let $\bar{w} = \Sigma^{-1}\sigma$ that maximizes the DR over long-short ptfs:

$$\forall w \in \Pi, \; DR(w) = DR(\bar{w})\varrho(\bar{w}, w) \implies \text{measure of realized } DR.$$

Realized Maximally Rho-presentativity



2278 US equity funds over Jan13-Mar17 : 80% of total US equity funds NAVs.

- The fund in **black** replicates the S&P500 index.
- Forward looking **constrained MDPs** and **EVW**.
- The fund in **red** targets the highest investable DR.

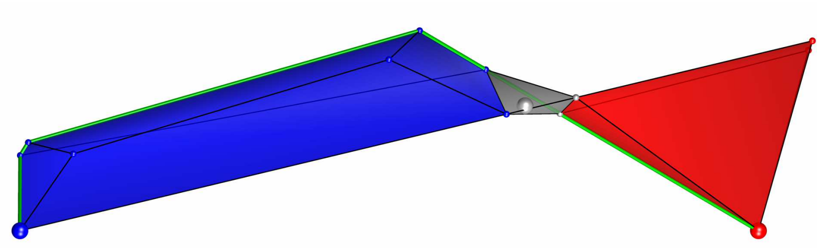
Conclusion

Key Concepts and Results

- The correlation spectrum provides an alternative and equivalent way of representing portfolios.
- Rho-presentative portfolios (ex: MV, MDP, ERC) allow investors to be positively exposed to all assets without being necessarily invested in all of them.
- Maximally Rho-presentative portfolios maximize under no particular constraint their aggregate exposure to all assets, as measured by a function f that is symmetric, concave and increasing.
- Such portfolios are long-only, and a basic characterization was provided, that is independent of f. They form a tiny subset of long-only portfolios, where we recovered well-known and possibly constrained investment strategies.
- This tiny set has attracted significant investments for more than a decade.

Key Applications

- The study of Maximally Rho-presentative portfolios leads to a unifying framework for constructing portfolios. It encompasses EW, EVW, ERC, constrained MV and MDP.
- We extend previous results of Jagannathan and Ma, by identifying explicitly the impact of maximum weight constraints on optimized portfolios such as the MV and MDP.
- We showed how to measure ex-post the Diversification and Rho-presentativity of funds without knowing their composition.
- Beyond their financial implications, these results may be useful in other fields where correlations are used to measure interactions.

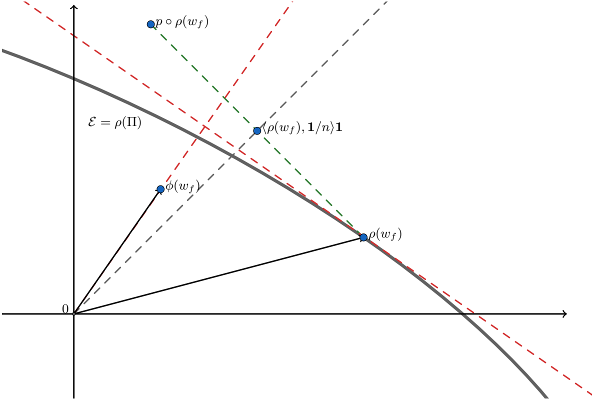


Thank you! Questions?

Sketch of Proof of the Main Theorem

Step 1: $\mathcal{R} \subset \Pi^+$: *exhibit* a feasible $w \in \Pi^+$ for $\operatorname{argmin}_{w \in \Pi^+, \rho(w) \succeq \rho(y)} 1$, relaxed to $\operatorname{argmin}_{w \succeq 0, \Sigma w \succeq \Sigma y} \sigma(w)$ (*constructive*).

Step 2: $w \in \mathcal{R} \implies \langle \phi(w)^{\uparrow}, \rho(w)^{\downarrow} \rangle = \langle \phi(w), \rho(w) \rangle$:
if not, $\exists p \in \mathfrak{S}_n$ with $\langle \phi(w), p \circ \rho(w) \rangle < \langle \phi(w), \rho(w) \rangle$.



Sketch of Proof of the Main Theorem

Step 3: $\langle \phi(w)^{\uparrow}, \rho(w)^{\downarrow} \rangle = \langle \phi(w), \rho(w) \rangle \implies w \in \mathcal{R}$:

$\forall w \in \Pi^{+}, z \mapsto f_w(z) := \langle \phi(w)^{\uparrow}, z^{\downarrow} \rangle$ is increasing, concave, sym and $\forall z \neq 0$,

$$f_w \circ \rho(z) \leq \langle \phi(w), \rho(z) \rangle \leq \langle \phi(w), \rho(w) \rangle = \langle \phi(w)^{\uparrow}, \rho(w)^{\downarrow} \rangle = f_w \circ \rho(w).$$

Step 4: $\lambda_{n-1}(\mathcal{R}) \leq \lambda_{n-1}\left(\Pi^{+\uparrow}\right)$ assuming $\sigma = \mathbf{1}$:
We have $\forall (w,p) \in \mathcal{R} \times \mathfrak{S}_n$, $p(w) \in \mathcal{R} \implies p(w) = w$, thanks to:

$$f_w \circ \rho(z) \leq \langle p(w), \rho(z) \rangle \leq \langle p(w), \rho(p(w)) \rangle = \langle p(w)^{\uparrow}, \rho(p(w))^{\downarrow} \rangle = f_w \circ \rho(p(w)).$$

Then, denoting $\Delta_p := \mathcal{R} \cap \left\{ w \in \mathbb{R}^n, p(w) = w^{\uparrow} \right\}$,

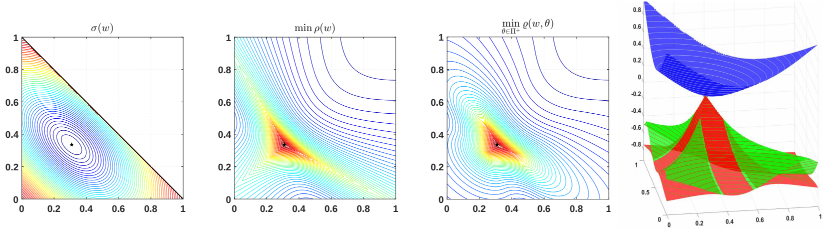
$$\lambda_{n-1}(\mathcal{R}) = \sum_{p \in \mathfrak{S}_n} \lambda_{n-1}(\Delta_p) = \sum_{p \in \mathfrak{S}_n} \lambda_{n-1}(p(\Delta_p)) \leq \lambda_{n-1}\left(\Pi^{+\uparrow}\right)$$

A Not-So-Typical Saddle-Point Problem

$$\Sigma = \begin{pmatrix} 1.0 & -0.3 & -0.4 \\ -0.3 & 1.0 & -0.5 \\ -0.4 & -0.5 & 1.0 \end{pmatrix} \succ 0$$

We obtained in our "Primal-Dual" framework that:

$$\min_{w \in \Pi^+} \sigma(w) = \max_{w \in \mathbb{R}^n \setminus \{0\}} \min \rho(w) = \max_{w \in \mathbb{R}^n \setminus \{0\}} \min_{\theta \in \Pi^+} \varrho(w, \theta).$$



- Solves an *a priori* difficult problem (not quasi concave-quasi convex).
- minimizing a quadratic form over Π^+ is equivalent to solve an unconstrained problem. Can we include more constraints?
- $\min_{\theta \in \Pi^+} \varrho(w, \theta) < \min \rho(w)$ may happen if w is not Rho-presentative.