AGGREGATION OF DYNAMIC PREFERENCES.

APPLICATION TO LONG-TERM YIELD CURVE MODELING.

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FINANCING LONG-TERM PROJECTS

- Modelling the long-term interest rates : a crucial issue for the financing of projects whose benefits will only be felt in the long term,
  - retirement savings
  - environmental projects
- The low interest rate policy is hardly compatible with the development of long-term projects.

What tools are available to the public authorities for assessing policies whose impact will only be felt several decades later?

- Traditional approaches, based on the theory of general equilibrium, are not always flexible and adaptive enough to apprehend the long-term
- The issue of the heterogeneity of economic actors is often downplayed in concrete applications

Equilibrium and representative agent

- Most of general equilibrium macroeconomic models are simplified by assuming that investors and/or firms could be described as a representative agent.
  - Agents may differ and act differently, but at equilibrium the sum of their choices is mathematically equivalent to the decision of one individual or many identical individuals.
  - The way that preferences of multiple agents aggregate at equilibrium is a difficult task
  - Even if each individual preference is modeled by a simple function, it is **unlikely that the aggregate utility could be reduced into a simple expression**
- Heterogeneity of investors is an unavoidable feature that should be taken into account. seminal paper by Dumas (1989), Cvitanic, Jouini et al. (2011), Abbot (2018).
- Existence of an equilibrium is often stated and studied in a **complete market setting**
- One key point for the existence of equilibrium is that **agents agree on the same pricing kernel**.

Consistent progressive utility

To go beyond the standard approach based on deterministic power utilities :

Consistent progressive utility

- To incorporate the possibility of changes in agent preferences over time, depending on the uncertain evolution of the economic or financial environment
- Musiela and Zariphopoulou (2007,2010) were the first to suggest to use instead of the classic criterion the concept of progressive dynamic utility
- The utility criterion must be
  - **adaptive and adjusted to the information flow**
  - **consistent** with respect to a given investment universe.
- Theoretical study of progressive utility (see El Karoui, Mrad (2013)) emphasizes the dependency of the optimal processes with respect to their initial conditions
  - useful to analyse the impact of the heterogeneity and of the market's wealth on the yield curve

Aggregation without equilibrium

How to describe globally the behavior and preferences of heterogeneous agents, even if no equilibrium exists?

- We start from the **weaker hypothesis of non arbitrage**, and we consider an **incomplete market**
- Starting point is the aggregate wealth of the economy, with a given repartition of the wealth among investors, which is not necessarily Pareto optimal.
- **Calibration approach** : How to derive a utility process for which the aggregate wealth is optimal?
- Construction of an aggregate progressive utility
  - market consistent,
  - aggregates the individual utility of the heterogeneous agents.
  - based on the aggregation of the pricing kernels of each investor

Outline

INVESTMENT UNIVERSE

Incomplete Itô market, defined on a filtered probability space $(\Omega, (\mathcal{F}_t), \mathbb{P})$ driven by a n-standard Brownian motion W.

Market Parameters

- d risky assets, $d \leq n$.
- $(r_t)_{t\geq 0}$, $(\eta_t)_{t\geq 0}$, $(\sigma_t)_{t\geq 0}$ adapted processes.
- $r_t \geq 0$ spot rate.
- η_t n-dimensional risk premium vector.
- σ_t volatility process $d \times n$.

Utility function

- u strictly concave, strictly increasing, non-negative function on $\mathbb{R}^+$,
- Continuous *marginal utility* u_x, satisfying the Inada conditions $\lim_{x\mapsto\infty} u_x(x) = 0$ and $\lim_{x\mapsto 0} u_x(x) = \infty$.
- Relative risk aversion coefficient $\mathrm{R}^{\mathrm{r}}_{\mathrm{A}}(u)(x) = -x u_{xx}(x)/u_x(x)$.
- Convention: small letters for deterministic utilities, capital letters for stochastic utilities.

Admissible portfolio

- Admissible strategy $\kappa_t := \sigma_t.\pi_t$
  with π_t : fractions of the wealth X_t invested in the risky assets.
- Constraints on the portfolio $\Rightarrow$ Incompleteness of the market.
  $\kappa_t \in \mathcal{R}_t$ where $\mathcal{R}_t$ adapted subvector spaces in $\mathbb{R}^n$.
- Self financing dynamics of wealth process with risky portfolio κ is given by

$$dX_t^\kappa = X_t^\kappa[r_t dt + \kappa_t(dW_t + \eta_t^{\mathcal{R}} dt)], \quad \kappa_t \in \mathcal{R}_t, \quad X_0^\kappa = x. \tag{1}$$

- $\mathscr{X}$:= set of wealth processes X^κ with admissible $\kappa \in \mathcal{R}$.

PRICING KERNEL

- Y^ν is called a pricing kernel if for any admissible wealth process X^κ

  the process $X_t^\kappa \, Y_t^\nu$ is a local martingale.

- differential decomposition of Y^ν

$$dY_t^\nu = Y_t^\nu[-r_t dt + (\nu_t - \eta_t^{\mathcal{R}}).dW_t], \quad \nu_t \in \mathcal{R}_t^\perp, \quad Y_0^\nu = y. \qquad (2)$$

- $\mathscr{Y}$:= the family of all pricing kernels Y^ν where $\nu \in \mathcal{R}^\perp$.

Definition of a consistent progressive utility

As in statistical learning, the utility criteria are dynamically adjusted given the family of test processes $\mathscr{X}$, also called the **learning set**.

- Progressive utilities **U**: adapted processes such that $\mathbb{P}$ as, for every $t \geq 0$, $x \to U(t,x)$ are standard utility functions.
- The utility **U** is said to be $\mathscr{X}$-consistent, if
  - for any admissible test process $X^\kappa \in \mathscr{X}$, the preference process $(U(t, X_t^\kappa))$ **is a non-negative supermartingale.**
  - there exists an optimal process $X^* := X^{\kappa^*} \in \mathscr{X}$, with $\kappa_t^* \in \mathcal{R}_t$, binding the constraint, in the sense that $(U(t, X_t^*))$ **is a martingale**.

Dual consistent progressive utility

The *consistency property* of the progressive utility **U** has a natural equivalent for dual progressive utility

- Dual utility $\widetilde{U}(t,y) = \sup_{x>0}\big(U(t,x) - yx\big)$.
- **U** is a consistent progressive utility with the class $\mathscr{X}$ if and only if its Fenchel transform $\widetilde{\mathbf{U}}$ is consistent with the class $\mathcal{Y}$ in the sense that $\widetilde{U}(t, Y_t)$ is a submartingale for any $Y \in \mathcal{Y}$, and there exists some $Y^* \in \mathcal{Y}$ (optimal pricing kernel) such that $\widetilde{U}(t, Y_t^*)$ is a martingale.
- the two optimal processes are related by the main identity

$$U_x(t, X_t^*(x)) = Y_t^*(u_x(x))$$

- General assumption that the utility random field **U** is a "regular" Itô random field with differential decomposition

$$dU(t,x) = \beta(t,x)dt + \gamma(t,x).dW_t \tag{3}$$

- $\beta(t,x)$ is the **drift random field**
- $\gamma(t,x)$ is the **multivariate diffusion random field**.

How to read on the local characteristics (β,γ) that the process $U(t,x)$ is a $\mathscr{X}$-consistent utility random field?

CONSISTENCY CHARACTERIZATION THROUGH A HJB CONSTRAINT

- The utility random field **U is $\mathscr{X}$-consistent if and only if**
  - The drift random field β satisfies the HJB-constraint, $d\mathbb{P} \times dt.a.s.$

$$\beta(t,x) = -U_x(t,x)xr_t + \frac{1}{2}U_{xx}(t,x)\|\sigma^*(t,x))\|^2.$$

  - The stochastic differential equation SDE$^{\mathcal{R}}(\sigma^*)$

$$\begin{cases} dX_t^* & = r_t X_t^* dt + \sigma^*(t,X_t^*)(dW_t + \eta_t^{\mathcal{R}} dt), \\ \sigma^*(t,x) & = -\frac{U_x(t,x)}{U_{xx}(t,x)}\left(\eta_t^{\mathcal{R}} + \frac{\gamma_x^{\mathcal{R}}(t,x)}{U_x(t,x)}\right) = x\kappa^*(t,x) \end{cases} \tag{4}$$

    admits a strong solution X^*, which is an optimal portfolio in the preference sense.
- In addition, the positive process $U_x(t, X_t^*(x))$ is the optimal pricing kernel $Y_t^*(u_x(x))$.

This HJB-characterization is necessary and sufficient condition to construct a consistent progressive utility from the optimal processes X^*, Y^*.

Outline

Aggregation of the initial utilities

- Different agents with characteristics represented by $(\mathbf{U}^\theta, m(d\theta))$:
  - $\mathbf{U}^\theta$ a consistent progressive utility
  - $m(d\theta)$ a weight
- Aggregation of the initial utilities
  - The θ-agent/class starts (at time 0) with a **proportion α^θ of the initial global wealth** x

$$u(x) = \int \frac{1}{\alpha^\theta} u^\theta(\alpha^\theta x)\ m(d\theta), \quad \int \alpha^\theta m(d\theta) = 1.$$

  - The marginal utility u_x of the global utility is the sum of the marginal utilities

$$u_x(x) = \int u_x^\theta(\alpha^\theta x)\ m(d\theta).$$

  - Dual relation using $u_x^{-1}(y) = -\tilde{u}_y(y)$:

$$y = \int y^\theta(y)\ m(d\theta), \quad y^\theta(y) = u_x^\theta(-\alpha^\theta \tilde{u}_y(y))$$

  - The relative risk aversion coefficient $\mathrm{R}_{\mathrm{A}}^{\mathrm{r}}(u)$ is a "probabilistic" mixture of the different risk aversion coefficients,

$$\mathrm{R}_{\mathrm{A}}^{\mathrm{r}}(u)(x) = \frac{-xu_{xx}(x)}{u_x(x)} = \int \mathrm{R}_{\mathrm{A}}^{\mathrm{r}}(u^\theta)(\alpha^\theta x) \frac{u_x^\theta(\alpha^\theta x)}{\int u_x^\theta(\alpha^\theta x)\ m(d\theta)}\ m(d\theta)$$

AGGREGATION OF THE OPTIMAL PROCESSES

- Aggregate wealth (X_t^*) defined as the weighted sum of the individual wealths $(X_t^{*,\theta})$,

$$X_t^*(x) := \int X_t^{*,\theta}(\alpha^\theta x)m(d\theta). \tag{5}$$

- Aggregate dual process $Y_t^*(u_x(x))$ is defined as a mixture of individual pricing kernels

$$Y_t^*(u_x(x)) := \int Y_t^{*,\theta}(u_x^\theta(\alpha^\theta x))m(d\theta) = \int Y_t^{*,\theta}(y^\theta(u_x(x)))m(d\theta). \tag{6}$$

- **Regularity of the aggregate processes** $X_t^*(x)$ **and** $Y_t^*(y)$ from the regularity of the individual processes $X_t^{*,\theta}(x)$ and $Y_t^{*,\theta}(y)$.
  - X^* is an admissible portfolio in $\mathscr{X}(x)$ issued from x

$$\begin{cases} dX_t^*(x) & = r_t X_t^*(x)dt + \phi^*(t,x).(dW_t + \eta_t^{\mathcal{R}}\, dt) \\ \phi^*(t,x) & := \int \phi^{*,\theta}(t,\alpha^\theta x))m(d\theta). \end{cases} \tag{7}$$

  - Y^* is an admissible pricing kernel in $\mathscr{Y}(y)$ issued from $y = u_x(x)$

$$\begin{cases} dY_t^*(y) & = -r_t Y_t^*(u_x(x))dt + \left(\psi^*(t,y) - Y_t^*(y)\eta_t^{\mathcal{R}}\right).dW_t. \\ \psi^*(t,y) & := \int \psi^{*,\theta}(t, y_x^\theta(y)))m(d\theta). \end{cases}$$

AGGREGATE UTILITY

- $X^*_\cdot(x) \in \mathscr{X}(x)$ and $Y^*_\cdot(y) \in \mathscr{Y}(y)$ are *increasing* monotonic processes $\rightarrow$ let $\mathcal{X}^*$ be the inverse flow of X^*

Main result

Construction of the aggregate utility from the optimal processes X^* and Y^*

- $U(t,x) = \int \int_0^x U_x^\theta(t, X_t^{*,\theta}(\alpha^\theta \mathcal{X}_t^*(z)))dz\, m(d\theta)$ is a consistent semimartingale progressive utility.
- with optimal primal and dual processes are $(X_t^*(x))$ and $\big(Y_t^*(u_x(x)) = U_x(t, X_t^*(x))\big)$
- The local characteristics of the aggregate utility are

$$\begin{cases} \gamma_x^{\mathcal{R}}(t,x) = -U_x(t,x)\eta_t^{\mathcal{R}} - U_{xx}(t,x)\phi^*(t, \mathcal{X}^*(t,x)). \\ \gamma_x^{\perp}(t,x) = \psi^*(t, u_x(\mathcal{X}^*(t,x))). \\ \beta(t,x) = -r_t x U_x(t,x) + \frac{1}{2} U_{xx}(t,x)||\phi^*(t, \mathcal{X}^*(t,x))||^2. \end{cases}$$

Aggregating power utilities

$\mathbf{U}^{(\theta)}$ power utilities with constant relative risk aversion coefficient θ ($0 < \theta < 1$).
The optimal processes are linear with respect to their initial conditions:

$$X_t^{*,(\theta)}(x) = x\bar{X}_t^{*,(\theta)}$$

- The aggregate marginal utility $U_x(t,x)$ is the deterministic aggregation of the power marginal progressive utilities with random repartition of the optimal wealth,

$$U_x(t,x) = \int U_x^{(\theta)}\Big(t, \frac{\alpha^{\theta}\bar{X}_t^{*,(\theta)}}{\bar{X}_t^{*}}x\Big)m(d\theta).$$

- The ratio $\bar{\alpha}_t^{\theta} = \frac{\alpha^{\theta}\bar{X}_t^{*,(\theta)}}{\bar{X}_t^{*}}$ is the stochastic ratio of the optimal wealths at time t.
- Aggregating power utilities provides a family of consistent progressive utilities which is more flexible, while benefiting from some interesting features of power utilities (such as tractability).

Outline

1. Investment universe and Consistent progressive utility
2. Aggregating multi-agents preferences
3. Application to the long term yield curve

Marginal utility indifference pricing

High illiquidity of the bond market for long maturities

- financial evaluation of zero-coupon bond : **marginal utility indifference pricing**
- link with the economic discount rate given by the **Ramsey rule**

Marginal utility indifference pricing

- This pricing is based on the optimal pricing kernel $Y^*(y)$ of the progressive dual utility $\widetilde{\mathbf{U}}$ of $\mathbf{U}$.
- The price at time t of some derivative ξ_T is given by

$$p^*_{t,T}(\xi_T)(y) = \mathbb{E}(\frac{Y^*_T(y)}{Y^*_t(y)}\xi_T|\mathcal{F}_t)$$

- Marginal utility Bond curve

$$B^*_t(T,y) = \mathbb{E}(\frac{Y^*_T(y)}{Y^*_t(y)}|\mathcal{F}_t)$$

$\Rightarrow$ the price depends on the global wealth x of the economy via the correspondence $u_x(x) = y$

Yield curve in aggregate economy

- Denoting $L_t^{*,\theta}(y) := L_t^{\nu^{*,\theta}}(y) = ye^{\int_0^t \vartheta_s^{*,\theta}(y)dW_s - \frac{1}{2}\int_0^t ||\vartheta_s^{*,\theta}(y)||^2 ds}$:
- In an aggregate economy
  - The **marginal utility bond curve** $B_t^*(T,y)$ is a normalized mixture of individual bond curves, based on the martingales $L_t^{*,\theta}$,

$$B_t^*(T,y) = \int B_t^{*,\theta}(T,y^\theta)\frac{L_t^{*,\theta}(y^\theta)}{\int L_t^{*,\theta}(y^\theta)m(d\theta)}\, m(d\theta). \tag{8}$$

  - The **marginal utility spot forward rates** $f_t^*(T,y)$ is a normalized mixture of individual spot forward rates curve based on the martingales $Y_t^{*,\theta}(y^\theta)B_t^{*,\theta}(T,y^\theta)$

$$f_t^*(T,y) = \int f_t^{*,\theta}(T,y^\theta)\frac{B_t^{*,\theta}(T,y^\theta)L_t^{*,\theta}(y^\theta)}{\int B_t^{*,\theta}(T,y^\theta)L_t^{*,\theta}(y^\theta)m(d\theta)}\, m(d\theta).$$

Indifference Bonds pricing for power utilities

- N **agents** with consistent **power utilities** characterized by their relative risk aversion parameters $\theta_1 < \cdots < \theta_N$
- Asymptotic behavior

$$\lim_{y \to 0} B_0^*(T, y) = B_0^{\theta_1}(T) \text{ and } \lim_{y \to +\infty} B_0^*(T, y) = B_0^{\theta_N}(T).$$

  - when the wealth tends to infinity, the aggregate zero-coupon price converges to the one priced by the less risk averse agent,
  - when the wealth tends to zero, it converges to the one priced by the more risk averse agent.

INDIVIDUAL YIELD CURVE FOR DIFFERENT VALUES OF THE RISK AVERSION

Yield curve $R_t^*(\delta) = -\frac{1}{\delta}\ln B_t^*(t+\delta) = \frac{1}{\delta}\int_0^\delta f_t^*(t+u)du$

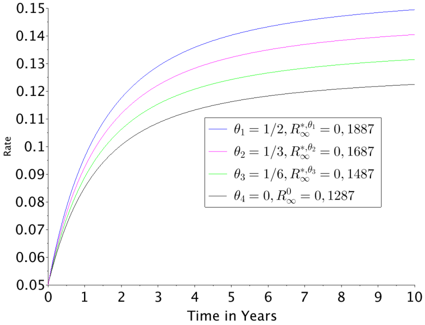

FIGURE: Individual yield curve $R_0^{*,\theta}(\delta)$ for different values of the risk aversion θ

Individual and aggregate yield curve spread

Spreads between the different rate curves and the market yield curve $R_0^0(\delta)$:
$spread = R_0^{*,\theta}(\delta) - R_0^0(\delta)$.

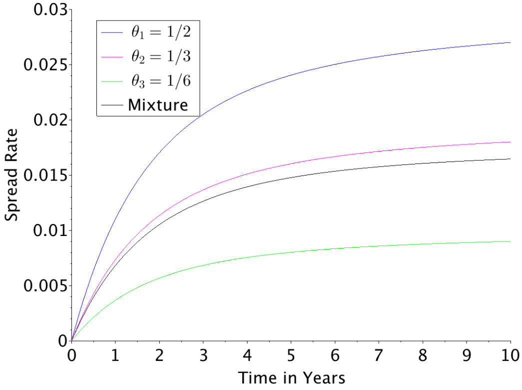

FIGURE: Individual and aggregate yield curve spread

AGGREGATE YIELD CURVE SPREAD DEPENDING ON THE WEALTH x

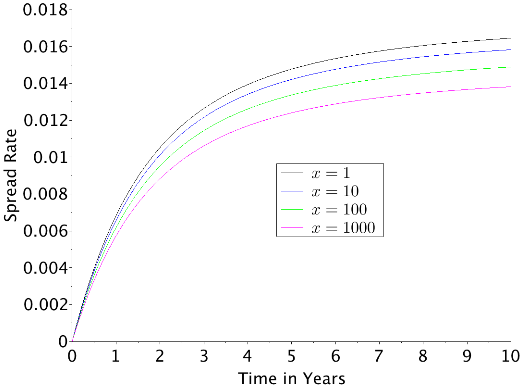

FIGURE: Aggregate yield curve spread depending on the wealth x

AGGREGATE YIELD CURVE SPREAD DEPENDING ON INITIAL PROPORTION PARAMETERS α

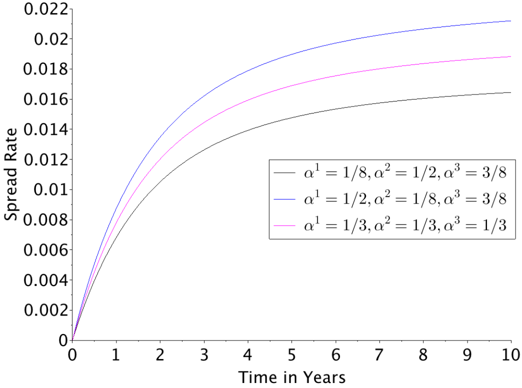

FIGURE: Aggregate yield curve spread depending on the initial proportion parameters α

Bibliography

[1] T. Abbot. General Equilibrium Under Convex Portfolio Constraints and Heterogeneous Risk Preferences. 2018

[2] F.P. Berrier, L.G.C. Rogers, and M.R. Tehranchi. A characterization of forward utility functions. Preprint, 2009

[3] Jaksa Cvitanic, Elyes Jouini, Semyon Malamud, and Clotilde Napp. Financial markets equilibrium with heterogeneous agents. Review of Finance, 16(1):285–321, 2011

[4] Bernard Dumas. Two-person dynamic equilibrium in the capital market. The Review of Financial Studies, 2(2):157–188, 1989.

[5] N. El Karoui, M. Mrad. An Exact Connection between two Solvable SDEs and a Non Linear Utility Stochastic PDEs, SIAM Journal on Financial Mathematics (2013).

[6] M. Musiela and T. Zariphopoulou. Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model. In Advances in mathematical finance, (2007).

[7] M. Musiela and T. Zariphopoulou. Investment and valuation under backward and forward dynamic exponential utilities in a stochastic factor model. Advances in mathematical finance, (2010).